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Abstract

Let $q \geq 1+\frac{2}{N}$. We prove that any positive solution of (E) $\partial_t u - \Delta u + u^q = 0$ in $\mathbb{R}^N \times (0,\infty)$ admits an initial trace which is a nonnegative Borel measure, outer regular with respect to the fine topology associated to the Bessel capacity $C_{\frac{2}{q},q'}$ in $\mathbb{R}^N$ ($q' = q/q-1$)) and absolutely continuous with respect to this capacity. If ν is a nonnegative Borel measure in $\mathbb{R}^N$ with the above properties we construct a positive solution u of (E) with initial trace ν and we prove that this solution is the unique σ-moderate solution of (E) with such an initial trace. Finally we prove that every positive solution of (E) is σ-moderate.

Contents

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1 Introduction

Let $q > 1$, $Q_T = \mathbb{R}^N \times (0, T)$ with $T > 0$ and $Q = \mathbb{R}^N \times (0, \infty)$. It is proved by Marcus and Véron [19] that for any positive function $u \in C^{2,1}(Q_T)$ solution of

$$\partial_t u - \Delta u + u^q = 0 \tag{1.1}$$

there exists a unique couple $(\mathcal{S}, \mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^N$ and μ a positive Radon measure on $\mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$ such that

$$\lim_{t\to 0} \int_{\mathcal{O}} u(x,t)dx = \infty \tag{1.2}$$

for all open set $\mathcal{O}$ of $\mathbb{R}^N$ such that $\mathcal{S} \cap \mathcal{O} \neq \emptyset$, and

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N} \zeta(x)d\mu(x) \qquad \forall \zeta \in C_0^\infty(\mathcal{R}). \tag{1.3}$$

To this couple $(\mathcal{S}, \mu)$ it is associated a unique outer Borel measure ν called *the initial trace* of u and denoted by $tr(u)$. The set $\mathcal{S}$ is *the singular set* of ν and the measure μ is *the regular set* of ν. Conversely, to any outer Borel measure ν we can associate its singular part $\mathcal{S}(\nu)$ which is a closed subset of $\mathbb{R}^N$ and its regular part μ_ν which is a positive Radon measure on $\mathcal{R}(\nu)$. We denote $\nu \approx (\mathcal{S}, \mu)$. When $1 < q < q_c := \frac{N+2}{N}$ Marcus and Véron [19] proved that the trace operator tr defines a one to one correspondence between the set $\mathcal{U}_+(Q_T)$ of positive solutions of (1.1) in Q_T and the set $\mathfrak{B}^{reg}(\mathbb{R}^N)$ of positive outer Borel measures in $\mathbb{R}^N$. This no longer the case if $q \geq q_c$ since not any closed subset of $\mathbb{R}^N$ (resp. any positive Radon measure) is eligible for being the singular set (resp. the regular part) of the the initial trace of some positive solution of (1.1). It is proved in [4] that the initial value problem

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u &= 0 \qquad && \text{in } Q \\ u(.,0) &= \mu && \text{in } \mathbb{R}^N \end{aligned} \tag{1.4}$$

where μ is a positive bounded Radon measure admits a solution if and only if μ satisfies

$$C_{\frac{2}{q},q'}(E) = 0 \Longrightarrow \mu(E) = 0 \quad \forall E \subset \mathbb{R}^N, E \text{ Borel}, \tag{1.5}$$

where $C_{\frac{2}{q},q'}$ stands for the Bessel capacity in $\mathbb{R}^N$ ($q' = q/(q-1)$). It is shown in [19] that this result holds even if μ is unbounded; this solution is unique and denoted u_μ. If G is a Borel subset of $\mathbb{R}^N$ we denote by $\mathfrak{M}_q(G)$ the set of Borel measures μ in G with the property that

$$C_{\frac{2}{q},q'}(E) = 0 \Longrightarrow \mu(E) = 0 \quad \forall E \subset G, E \text{ Borel}, \tag{1.6}$$

In the same article it is proved that a necessary and sufficient condition in order $\nu \approx (\mathcal{S}, \mu)$ to be the initial trace of a positive solution of (1.1) is

$$\mu \in \mathfrak{M}_q(\mathcal{R}) \tag{1.7}$$

and

$$\mathcal{S} = \partial_\mu \mathcal{S} \bigcup \mathcal{S}^* \tag{1.8}$$

where

$$\partial_\mu \mathcal{S} = \{z \in \mathcal{S} : \mu(B_r(z) \bigcap \mathcal{S}) = \infty\,,\ \forall r > 0\} \tag{1.9}$$

and

$$\mathcal{S}^* = \{z \in \mathcal{S} : C_{\frac{2}{q},q'}((B_r(z) \bigcap \mathcal{S}) > 0\,,\ \forall r > 0\}. \tag{1.10}$$

The meaning of (1.8) is that the singular set is created either by the local unboundedness of the Radon measure or because the singular set is localy non-removable. Furthermore the solution which is constructed is the maximal solution with initial trace $(\mathcal{S}, \mu)$.

A striking result due to Le Gall [15] shows that if $q = 2$ and $N \geq 2$, a positive solution of (1.1) is not uniquely determinef by its initial trace $\nu \approx (\mathcal{S}, \mu)$ if $\mathcal{S} \neq \emptyset$. The results is actually extended to any $q \geq q_c$ in [19]. The main point in this counter-example relies on the construction of a positive solution u of (1.1) with a singular set $\mathcal{S} = \mathbb{R}^N$, with a blow-up set at $t = 0$ which the union of a countable of closed balls $\overline{B}_{\epsilon_n}(a_n)$ where $\{a_n\}$ is a dense set in $\mathbb{R}^N$ and the ϵ_n are chosen small enough so that $u(0,1) \leq \alpha$ for some $\alpha > 0$ fixed. If $U_{\overline{B}_{\epsilon_n}(a_n)}$ denotes the solution with initial trace $(\overline{B}_{\epsilon_n}(a_n), 0)$, then $U_{\overline{B}_{\epsilon_n}(a_n)}(0,1) \leq C(\epsilon_n)$ with $\lim_{\epsilon\to 0} C(\epsilon)$. This is a consequence of the supercriticality assumption and the estimates in [22]. The solution u is constructed between a sub-solution and a super-solution

$$\sup_n \{U_{\overline{B}_{\epsilon_n}(a_n)}\} \leq u \leq \sum_{n=0}^{\infty} U_{\overline{B}_{\epsilon_n}(a_n)}, \tag{1.11}$$

the right-hand side being chosen so that $\sum_{n=0}^{\infty} C(\epsilon_n) \leq \alpha$. Denoting $E = \cup_n \overline{B}_{\epsilon_n}(a_n)$, then $|E| < \infty$ and u satisfies

$$\lim_{t\to 0} u(x,t) = 0 \qquad \forall x \in \mathbb{R}^N \setminus E \text{ where } |E| < \infty, \tag{1.12}$$

and

$$\lim_{t\to 0} t^{\frac{1}{q-1}} u(x,t) = c_q = (q-1)^{\frac{1}{1-q}} \quad \text{ uniformly for } x \in K \subset \bigcup_n B_{\epsilon_n}(a_n),\ K \text{ compact.} \tag{1.13}$$

Thus (1.2) holds for any nonempty open set $\mathcal{O} \subset \mathbb{R}^N$. This counter-example points out that the trace process associated to averaging a positive solution u of (1.1) on open sets and letting $t \to 0$ is not sharp enough to distinguish among solutions; this process is now called the *rough trace*. This is why the introduction of a finer averaging appears to be needed. This finer averaging method is constructed by using the *fine topology* associated to the capacity $C_{\frac{2}{q},q'}$. It will lead us to the notion of precise trace.

A similar approach has been carried out if one considers the boundary trace problem for the positive solutions of the elliptic equation

$$-\Delta u + |u|^{q-1} u = 0 \qquad \text{in } \Omega \tag{1.14}$$

where Ω is a bounded C^2 domain in $\mathbb{R}^N$ $(N \geq 2)$ and $q > 1$. The boundary trace is defined in a somewhat similar way as the initial trace, by considering the limit in the weak sense of measures, of the restriction of u to the set $\Sigma_\epsilon := \{x \in \Omega : \text{dist}\,(x, \Omega^c = \epsilon)\}$, when $\epsilon \to 0$. The boundary trace $tr_{\partial\Omega}(u)$ is a uniquely determined outer regular Borel measure on $\partial\Omega$, with singular part $\mathcal{S}$, a closed subset of $\partial\Omega$ and regular part μ, a positive Radon measure on $\mathcal{R} = \partial\Omega \setminus \mathcal{S}$. This equation possesses a critical exponent $q_e = (N+1)/(N-1)$. The main contributions which lead to a complete picture of the boundary trace problem over a period of twenty years are due to Gmira and Véron [11], Le Gall [13], [14], Dynkin and Kuznetsov [5],[6], [7] [8], [9],[12], Marcus and Véron [17],[18],[20],[21],[23], [22], [16], and Mselati [24]. These contributions can be summarized as follows:

(i) If $1 < q < q_e$ the boundary trace operator establishes a one to one correspondence between the set $\mathcal{U}_+(\Omega)$ of positive solutions of (1.14) and the set of positive outer regular Borel measures on $\partial\Omega$.

(ii) If $q \geq q_e$ the boundary value problem

$$\begin{aligned} -\Delta u + |u|^{q-1}u &= 0 \qquad && \text{in } \Omega \\ u &= \mu && \text{in } \partial\Omega \end{aligned} \tag{1.15}$$

where μ is a positive Radon measure on $\partial\Omega$ admits a solution (always unique) if and only if

$$C_{\frac{2}{q},q'}(E) = 0 \Longrightarrow \mu(E) = 0 \quad \forall E \subset \partial\Omega, E \text{ Borel}, \tag{1.16}$$

where $C_{\frac{2}{q},q'}$ is the Bessel capacity in $\mathbb{R}^{N-1}$.

(iii) If $q \geq q_e$, a outer regular Borel measure $\nu \approx (\mathcal{S}, \mu)$ on $\partial\Omega$ is the boundary trace of a positive solution of (1.14) if and only if

$$C_{\frac{2}{q},q'}(E) = 0 \Longrightarrow \mu(E) = 0 \quad \forall E \subset \mathcal{S}, E \text{ Borel},$$

and (1.8) holds with (1.9) and (1.10) where the capacity is relative to dimension N-1.

(iv) If $q \geq q_e$ a solution is not uniquely determined by its boundary trace whenever $\mathcal{S} \neq \emptyset$.

However in [23] Marcus and Véron have defined a notion of *precise trace* for the case $q \geq q_e$ with the following properties,

(v) If we denote by $\mathfrak{T}_q$ the fine topology of $\partial\Omega$ associated with the $C_{\frac{2}{q},q'}$-capacity, there exists a $\mathfrak{T}_q$-closed subset $\mathcal{S}_q$ of $\partial\Omega$ such that for every $z \in \mathcal{S}_q$

$$\lim_{\epsilon \to 0} \int_\Xi u(\epsilon, \sigma) dS = \infty \tag{1.17}$$

for every $\mathfrak{T}_q$-open neighborhood Ξ of z where $(r, \sigma) \in [0, \epsilon_0] \times \partial\Omega$ are the flow coordinates near $\partial\Omega$, and for every $z \in \mathcal{R}_q := \partial\Omega \setminus \mathcal{S}_q$, there exists a $\mathfrak{T}_q$-open neighborhood Ξ of z such that

$$\limsup_{\epsilon \to 0} \int_\Xi u(\epsilon, \sigma) dS < \infty. \tag{1.18}$$

(vi) There exists a nonnegative Borel measure μ on $\mathcal{R}_q$, outer regular for the $\mathfrak{T}_q$-topology, such that

$$\lim_{\epsilon\to 0} u^{\Xi}_{\epsilon} = u_{\mu\chi_{\Xi}} \qquad \text{locally uniformly in } \Omega, \tag{1.19}$$

where u^{Ξ}_{ϵ} is the solution of

$$\begin{array}{rll} -\Delta v + |v|^{q-1}v = 0 & & \text{in } \Omega_\epsilon := \{x \in \Omega : \mathrm{dist}\,(x, \partial\Omega) > \epsilon\} \\ v = u(\epsilon, .)\chi_{\Xi} & & \text{in } \Sigma_\epsilon = \partial\Omega_\epsilon. \end{array} \tag{1.20}$$

The couple $(\mathcal{S}_q, \mu)$ is uniquely determined and it is called the precise boundary trace of u. It can also be represented by a Borel measure with the $\mathfrak{T}_q$-outer regularity. It is denoted by $tr^q_{\partial\Omega}(u)$.

Concerning uniqueness Dynkin and Kuznetsov introduced in [9] the notion of σ-moderate solutions, which are elements u of $\mathcal{U}_+(\Omega)$ with the property that there exists an increasing sequence $\{\mu_n\}$ of nonnegative Radon measures on $\partial\Omega$ such that $u_{\mu_n} \to u$ when $n \to \infty$. In [23] Marcus and Véron proved that a σ-moderate positive solution of (1.14) is uniquely determined by its precise boundary trace. This precise trace is essentially the same, up to a set of zero $C_{\frac{2}{c},q'}$-capacity, as the *fine trace* that Dynkin and Kuznetsov introduced in [9] using probabilistic tools such as the Brownian motion; however their construction is only valid in the range $(1, q]$ of values of q. Finally, in [16], Marcus proved that any positive solution is σ-moderate. Notice that this result was already obtained by Mselati [24] in the case $q = 2$ and then by Dynkin [6] for $q_e \le q \le 2$ by using a combination of analytic and probabilistic techniques.

In this article we define a notion of *precise initial trace* for positive solutions of (1.1) associated to the $\mathfrak{T}_q$-topology, which denotes the $C_{\frac{2}{q},q'}$ fine topology of $\mathbb{R}^N$. We denote by $\mathbb{H}[.]$ the heat potential in Q expressed by

$$\mathbb{H}[\xi](x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}}\int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}}\xi(y)dy, \tag{1.21}$$

for all $\xi \in L^1(\mathbb{R}^N)$. We define the *singular set* of $u \in \mathcal{U}_+(Q_T)$ as the set of $z \in \mathbb{R}^N$ such that for any $\mathfrak{T}_q$-open neighborhood $\mathcal{O} \subset \mathbb{R}^N$ of z, there holds

$$\iint_{Q_T} \mathbb{H}[\chi_{\mathcal{O}}]u^q dxdt = \infty. \tag{1.22}$$

The singular set, denoted by $\mathcal{S}_q = \mathcal{S}_q(u)$, is $\mathfrak{T}_q$-closed. The regular set is $\mathcal{R}_q := \mathbb{R}^N \setminus \mathcal{S}_q$; it is $\mathfrak{T}_q$-open. If $z \in \mathcal{S}_q$ and $\mathcal{O} \subset \mathbb{R}^N$ is a $\mathfrak{T}_q$-open neighborhood of z such that

$$\iint_{Q_T} \mathbb{H}[\chi_{\mathcal{O}}]u^q dxdt < \infty, \tag{1.23}$$

then for any $\eta \in L^\infty \cap W^{\frac{2}{q},q'}(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support contained in $\mathcal{O}$ there exists

$$\lim\nolimits_{t\to 0}\int_{\mathbb{R}^N} u(x,t)(\eta(x))^{2q'}dx := \ell_{\mathcal{O}}(\eta). \tag{1.24}$$

As a consequence there exists a positive Borel measure μ on $\mathcal{R}_q$, outer regular for the $\mathfrak{T}_q$-topology, such that for $\mathfrak{T}_q$-open subset $\Xi \subset \mathcal{R}_q$ there holds

$$\lim\nolimits_{\epsilon\to 0} u_{\epsilon,\chi_{\Xi}}(.,t) = u_{\chi_{\Xi}\mu} \tag{1.25}$$

where u_{ϵ,χ_Ξ} is the solution of

$$\begin{array}{ll} \partial_t v - \Delta v + |v|^{q-1} v = 0 & \text{in } Q^\epsilon := \mathbb{R}^N \times (\epsilon, \infty) \\ v(.,\epsilon) = \chi_\Xi & \text{in } \mathbb{R}^N. \end{array} \tag{1.26}$$

The set $(\mathcal{S}_q, \mu)$ is called the *precise initial trace of* u and denoted by $\mathrm{tr}^c(u)$. To this set we can associate a Borel measure ν on $\mathbb{R}^N$. It is absolutely continuous with respect to the $C_{\frac{2}{q},q'}$-capacity in the following sense

$$\forall Q \subset \mathbb{R}^N, \mathfrak{T}_q\text{- open }, \forall A \subset \mathbb{R}^N,\ A \text{ Borel }, C_{\frac{2}{q},q'}(A) = 0 \Longrightarrow \mu(Q \setminus A) = \mu(Q). \tag{1.27}$$

It is also outer regular with respect to the $\mathfrak{T}_q$-topology in the sense that for every Borel set $E \subset \mathbb{R}^N$

$$\mu(E) = \inf\{\mu(Q) : Q \supset E,\ Q\ \mathfrak{T}_q\text{- open }\} = \sup\{\mu(K) : K \subset E,\ K \text{ compact }\}. \tag{1.28}$$

A measure with the above properties is called $\mathfrak{T}_q$*-perfect.* Similarly to Dynkin, we say that a positive solution u of (1.1) is σ-moderate if the exists an increasing sequence $\{\mu_n\}$ of nonnegative Radon measures in $\mathbb{R}^N$ such that $u_{\mu_n} \to u$ when $n \to \infty$. It is proved in [22] that if $F \subset \mathbb{R}^N$ is a closed subset, the maximal solution U_F with initial trace $(F, 0)$ coincides with the maximal σ-moderate solution V_F with the same trace and which is defined by

$$V_F = \sup\{u_\mu : \mu \in \mathfrak{M}_q(\mathbb{R}^N), \mu(F^c) = 0\}. \tag{1.29}$$

It is indeed σ-moderate. Following Dynkin we define an addition among the elements of $\mathcal{U}_+(Q_T)$ by

$$\forall (u,v) \in \mathcal{U}_+(Q_T) \times \mathcal{U}_+(Q_T),\ u \oplus v \text{ is the largest element of } \mathcal{U}_+(Q_T) \text{ dominated by } u + v. \tag{1.30}$$

The main results of this article are the following

Theorem A. *If ν is a $\mathfrak{T}_q$-perfect measure with singular part $\mathcal{S}_q$ and regular part μ on $\mathcal{R}_q$ then $u_\mu \oplus U^{\mathcal{S}_q}$ is the only σ-moderate element of $\mathcal{U}_+(Q)$ with precise trace ν.*

In order to extend Marcus's result we need a parabolic counterpart of Ancona's characterization of positive solutions of Schrödinger equation with singular potential [1]. We prove a representation theorem valid for any positive solution of

$$\partial_t u - \Delta u + V(x,t) u = 0 \qquad \text{in } Q, \tag{1.31}$$

where V is a Borel function which satisfies, for some $c \geq 0$,

$$0 \leq V(x,t) \leq \frac{c}{t} \qquad \text{for almost all } (x,t) \in Q. \tag{1.32}$$

Let T be fixed and let ψ be defined by

$$\psi(x,t) = \int_t^T \int_{\mathbb{R}^N} \frac{1}{(4\pi(s-t))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(s-t)}} V(y,s)\,dy\,ds \qquad \text{in } Q_T.$$

Theorem B. *There exists a kernel* Γ *defined in* $Q_T \times Q_T$ *satisfying*

$$c_1 \frac{e^{-a_1 \frac{|x-y|^2}{s-t}}}{(t-s)^{\frac{N}{2}}} \leq \Gamma(x,t,y,s) \leq c_2 \frac{e^{-a_2 \frac{|x-y|^2}{s-t}}}{(t-s)^{\frac{N}{2}}} \quad \forall (x,t),(y,s) \in Q_T \times Q_T \text{ with } s \leq t. \tag{1.33}$$

where the a_j *and* c_j *are positive contants depending on* T *and* V*, such that for any positive solution* u *of (1.31), there exists a positive Radon measure* μ *in* $\mathbb{R}^N$ *such that*

$$u(x,t) = e^{\psi(x,t)} \int_{\mathbb{R}^N} \Gamma(x,t,y,0)d\mu(y) \qquad \text{for almost all } (x,t) \in Q_T. \tag{1.34}$$

The next result, combined with Theorem A, shows that in the case $q \geq q_c$ the precise trace operator realizes a one to one correpondence between the set of positive solutions of (1.1) and the set of $\mathfrak{T}_q$-perfect Borel measures in $\mathbb{R}^N$.

Theorem C *Any positive solution of (1.1) is* σ*-moderate.*

Several proofs in this work are transposition to the parabolic framework of the constructions performed in [23] and [16]. However, for the sake of completeness and due to the technicalities involved, we kept many of them, sometimes under an abriged form.

2 The $\mathfrak{T}_q$-fine topology

We assume that $q \geq 1 + \frac{2}{N}$ and set $q' = \frac{q}{q-1}$. We recall that a set $E \subset \mathbb{R}^N$ is $(\frac{2}{q}, q')$-*thin* at a point a if

$$\int_0^1 \left(\frac{C_{\frac{2}{q},q'}(E \cap B_s(a))}{s^{N-\frac{2}{q-1}}} \right)^{q-1} \frac{ds}{s} < \infty. \tag{2.35}$$

If the value of the above integral is infinite, the set E is called $(\frac{2}{q}, q')$-*thick* at a. A set U is a $(\frac{2}{q}, q')$-fine neighborhood of one of its point a if U^c is thin at a. It is $(\frac{2}{q}, q')$-*finely open*, if U^c is thin at any point $a \in U$. It is $(\frac{2}{q}, q')$-*finely closed* if it complement is $(\frac{2}{q}, q')$- finely open. For simplicity we will denote by $\mathfrak{T}_q$ the $(\frac{2}{q}, q')$-*fine topology* associated to these notions (see [2, Chap 6] for a thorough discussion of these notions). We say that a set $E \subset I\!R^N$ is $\mathfrak{T}_q$-open (resp $\mathfrak{T}_q$-closed) if it is open (resp. closed) in the $\mathfrak{T}_q$-topology.

Notation 2.1 *Let* $A,\ B \subset I\!R^N$.
a) A *is* $\mathfrak{T}_q$*-essentially contained in* B*, denoted* $A \subset^q B$*, if*

$$C_{\frac{2}{q},\, q'}(A \setminus B) = 0.$$

b) The sets $A,\ B$ *are* $\mathfrak{T}_q$*-equivalent, denoted* $A \sim^q B$*, if*

$$C_{\frac{2}{q},\, q'}(A \Delta B) = 0.$$

c)The $\mathfrak{T}_q$*-closure of a set* A *is denoted by* $\widetilde{A}$*. The* $\mathfrak{T}_q$*-interior of* A *is denoted by* $A^\diamond$*.*
d) Given $\varepsilon > 0$*,* A^ε *denotes the* $\varepsilon-$*neighbourhood of* A *for the standard Euclidean distance in*

$\mathbb{R}^N$
e) The set of $\mathfrak{T}_q$-thick points of A is denoted by $b_q(A)$. The set of $\mathfrak{T}_q$-thin points of A is denoted by $e_q(A)$.

$$A \text{ is } \mathfrak{T}_q\text{-open} \Leftrightarrow A \subset e_q(A^c), \qquad B \text{ is } \mathfrak{T}_q \text{ closed} \Leftrightarrow b_q(B) \subset B.$$

Consequently,

$$\widetilde{A} = A \bigcup b_q(A), \qquad A^{\diamond} = A \cap e_q(A^c).$$

The capacity $C_{\frac{2}{q},\,q'}$ possesses the Kellogg property (see [2, Cor. 6.3.17]), namely,

$$C_{\frac{2}{q},\,q'}(A \setminus b_q(A)) = 0. \tag{2.36}$$

Therefore

$$A \subset^q b_q(A) \sim^q \tilde{A},$$

but, in general, $b_q(A)$ does not contain A.

Proposition 2.2 *(i) If Q is a $\mathfrak{T}_q$-open, then $e_q(Q^c)$ is the largest $\mathfrak{T}_q$-open set that is $\mathfrak{T}_q$-equivalent to Q.*
(ii) If F is a $\mathfrak{T}_q$-closed then $b_q(F)$ is the smallest $\mathfrak{T}_q$-closed set that is $\mathfrak{T}_q$-equivalent to F.

The proof is [23, Prop. 2.1]. We collect below several facts concerning the $\mathfrak{T}_q$-topology that are used throughout the paper.

Proposition 2.3 *Let $q \geq 1 + \frac{2}{N}$.*
i) Every $\mathfrak{T}_q$-closed set is $\mathfrak{T}_q$-quasi closed ([2, Prop 6.4.13]).
ii)If E is $\mathfrak{T}_q$-quasi closed then $E \sim^q \widetilde{E}$ ([2, Prop 6.4.12]).
iii)A set E is $\mathfrak{T}_q$-quasi closed if and only if there exists a sequence $\{E_m\}$ of closed subsets of E such that $C_{\frac{2}{q},\,q'}(E \setminus E_m) \to 0$ ([2, Prop. 6.4.9]).
iv) There exists a positive constant c such that, for every set E,

$$C_{\frac{2}{q},\,q'}(\widetilde{E}) \leq c C_{\frac{2}{q},\,q'}(E),$$

([2, Prop 6.4.11]).
v) If E is $\mathfrak{T}_q$-quasi closed and $F \sim^q E$ then F is $\mathfrak{T}_q$-quasi closed.
vi) If $\{E_i\}$ is an increasing sequence of arbitrary Borel sets then

$$C_{\frac{2}{q},\,q'}(\bigcup E_i) = \lim_{i\to\infty} C_{\frac{2}{q},\,q'}(E_i).$$

vii) If $\{K_i\}$ is a decreasing sequence of compact sets then

$$C_{\frac{2}{q},\,q'}(\bigcap K_i) = \lim_{i\to\infty} C_{\frac{2}{q},\,q'}(K_i).$$

viii) Every Suslin set and, in particular, every Borel set E satisfies

$$\begin{aligned} C_{\frac{2}{q},\,q'}(E) &= \inf\{C_{\frac{2}{q},\,q'}(G) :\ E \subset G,\ G \text{ open}\} \\ &= \sup\{C_{\frac{2}{q},\,q'}(K) :\ K \subset E,\ K \text{ compact}\}. \end{aligned}$$

For the last three statements see [2, Sec. 2.3]. Statement (v) is an easy consequence of [2, Prop. 6.4.9]. However note that this assertion is no longer valid if "$\mathfrak{T}_q$-quasi closed" is replaced by "$\mathfrak{T}_q$-closed." Only the following weaker statements holds:

If E is $\mathfrak{T}_q$-closed and A is a set such that $C_{\frac{2}{q},\,q'}(A) = 0$ then $E \cup A$ is $\mathfrak{T}_q$-closed.

The next corollary is an easy consequence of (iii).

Corollary 2.4 *A set E is $\mathfrak{T}_q$-quasi closed if and only if there exists a sequence $\{E_m\}$ of $\mathfrak{T}_q$-quasi closed subsets of E such that $C_{\frac{2}{q},\,q'}(E \setminus E_m) \to 0$.*

Definition 2.5 *Let E be a $\mathfrak{T}_q$-quasi closed set. An increasing sequence $\{E_m\}$ of closed subsets of E such that $C_{\frac{2}{q},\,q'}(E \setminus E_m) \to 0$ is called a $\mathfrak{T}_q$-stratification of E.*
(i) We say that E_m is a proper $\mathfrak{T}_q$-stratification of E if

$$C_{\frac{2}{q},\,q'}(E_{m+1} \setminus E_m) \leq \frac{1}{2^{m+1}}.$$

(ii) If V is a $\mathfrak{T}_q$-open set such that $C_{\frac{2}{q},\,q'}(E \setminus V) = 0$ we say that V is a $\mathfrak{T}_q$-quasi neighborhood of E.

The following separation statement is valid in any locally compact metric space.

Lemma 2.6 *Let K be a closed subset of an open set A. Then there exists an open set G such that*

$$K \subset G \subset \overline{G} \subset A.$$

Proof. Let $x \in K$. We set $B_n = B_n(x)$; $n \in \mathbb{N}$ and $K_n = \overline{B}_n \cap K$. Since K_n is compact, we can easily show that there exists a decreasing sequence $\{\varepsilon_n\}$ converging to 0 such that $K_n^{\varepsilon_n} \subset \overline{K_n^{\varepsilon_n}} \subset A$. Now we have

$$\bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_n}{2}} \subset \bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_n}{2}}} \subset \bigcup_{n=1}^{\infty} K_n^{\varepsilon_n} \subset A.$$

If we prove that the set

$$\bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_n}{2}}}$$

is closed then the proof follows with $G = \bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_n}{2}}$. We will prove it by contradiction. We assume that there exists a sequence $x_n \in \bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_n}{2}}}$ such that $x_n \to x$ and $x \notin \bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_n}{2}}}$. We have $x_1 = x_{n_1}$ such that $dist(x_{n_1}, K) = \inf\{|x_{n_1} - y| : \ y \in K\} \leq \frac{\varepsilon_1}{2}$. Also we assert that there exists x_{n_2} such that $dist(x_{n_2}, K) \leq \frac{\varepsilon_2}{2}$. Indeed, If this is not valid then $\forall n \in \mathbb{N}$ we have $\frac{\varepsilon_2}{2} < dist(x_n, K) \leq \frac{\varepsilon_1}{2}$, which implies $x \in K_1$. Thus we have clearly a contradiction. Inductively, we can construct a subsequence $\{x_{n_k}\}$ such that $dist(x_{n_k}, K) \leq \frac{\varepsilon_k}{2}$, $\forall k \in \mathbb{N}$. If we send k to infinite, we reach to a contradiction, since we would have $dist(x, K) = 0$ and using the fact that K is closed, we would obtain that $x \in K$. $\square$

In the framework of the $\mathfrak{T}_q$-topology, the preceding result admits the following counterpart.

Lemma 2.7 *Let E be a $\mathfrak{T}_q$-closed set. Then:*
(i) Let D be an open set such that $C_{\frac{2}{q},q'}(E \setminus D^c) = 0$. Then there exists an open set O such that

$$E \subset^q O \subset \widetilde{O} \subset^q D. \tag{2.37}$$

(ii) Let D be a $\mathfrak{T}_q$-open set such that $E \subset^q D$. Then there exists a $\mathfrak{T}_q$-open set O such that (2.37) holds.

Proof. (i) Since $E \cap D \sim^q E$, $E \cap D$ is $\mathfrak{T}_q$-quasi closed, (see the discussion of the quasi topology in [2, sec. 6.4]). Thus there exists a proper $\mathfrak{T}_q$-stratification of $E \cap D$, say $\{E_m\}$ and $E \sim^q E' = \bigcup_{i=1}^{\infty} E_i$. If E' is closed the result follows by Lemma 2.6. We assume that E' is not closed. Thus, we can assume without loss of generality that

$$E_{m+1} \setminus E_m \neq \emptyset \qquad \forall m \in \mathbb{N}.$$

We set $E'_m = G$, where G is the open set of Lemma 2.6 with $K = E_m$ and $A = D$. Now since $C_{\frac{2}{q},q'}(E_m \setminus E_{m-1}) < \frac{1}{2^{m+1}}$, there exists an open set $D_m \supset E_m \setminus E_{m-1}$; $m \geq 2$, such that $C_{\frac{2}{q},q'}(D_m) < \frac{1}{2^m}$. Also we set $D_1 = E'_1$. Also we have by Lemma (2.6),

$$D_m \cap E_m \subset \widetilde{D_m \cap E_m} \subset \widetilde{E_m} \subset D \qquad \forall m \in \mathbb{N}.$$

Also, since $E' = E_1 \cup \bigcup_{m=2}^{\infty}(E_m \setminus E_{m-1})$ we have that

$$E' \subset \bigcup_{m=1}^{\infty} D_m \cap E'_m \subset \bigcup_{m=1}^{\infty} \widetilde{D_m \cap E'_m} \subset D.$$

Thus, it is enough to prove that the set $\bigcup_{m=1}^{\infty} \widetilde{D_m \cap E'_m}$ is $\mathfrak{T}_q$-quasi closed. Indeed, for each $n > 1$, we have

$$\begin{aligned} C_{\frac{2}{q},q'}\left(\bigcup_{m=1}^{\infty} \widetilde{D_m \cap E'_m} \setminus \bigcup_{m=1}^{n} \widetilde{D_m \cap E'_m}\right) &\leq C_{\frac{2}{q},q'}\left(\bigcup_{m=n+1}^{\infty} \widetilde{D_m \cap E'_m}\right) \leq \sum_{m=n+1}^{\infty} C_{\frac{2}{q},q'}(\widetilde{D_m}) \\ &\leq c \sum_{m=n+1}^{\infty} C_{\frac{2}{q},q'}(D_m) \leq c \sum_{m=n+1}^{\infty} 2^{-m}. \end{aligned}$$

And the result follows by Corollary 2.4, since $\bigcup_{m=1}^{n} \widetilde{D_m \cap E'_m}$ is $\mathfrak{T}_q$-quasi closed.
The proof of (ii) is same as in [23, Lemma 2.4 (ii)]. $\square$

Lemma 2.8 *(I) Let E be a $\mathfrak{T}_q$-closed set and $\{E_m\}$ a proper $\mathfrak{T}_q$-stratification for E. Then there exists a decreasing sequence of open sets $\{Q_j\}$ such that $\cup E_m := E' \subset Q_j$ for every $j \in \mathbb{N}$ and*

(i) $\cap_j Q_j = E'$, $\widetilde{Q}_{j+1} \subset^q Q_j$,
(ii) $\lim_{j\to\infty} C_{\frac{2}{q},\, q'}(Q_j) = C_{\frac{2}{q},\, q'}(E)$.
(II) If A is a $\mathfrak{T}_q$-open set, there exists a decreasing sequence of open sets $\{A_m\}$ such that

$$A \subset \bigcap_m A_m =: A', \qquad C_{\frac{2}{q},\, q'}(A_m \setminus A') \to 0 \ \textit{as}\ m \to \infty, \quad A \sim^q A'.$$

Furthermore there exists an increasing sequence of closed sets $\{F_j\}$ such that $F_j \subset A'$ and
(i) $\cup F_j = A'$, $\qquad F_j \subset^q F^{\diamond}_{j+1}$
(ii) $C_{\frac{2}{q},\, q'}(F_j) \to C_{\frac{2}{q},\, q'}(A')$ *as* $j \to \infty$.

Proof. Let $\{D_j\}$ be a decreasing sequence of open sets such that $D_j \supset E$, $\forall j \in \mathbb{N}$ and

$$\lim_{j\to\infty} C_{\frac{2}{q},\, q'}(D_j) = C_{\frac{2}{q},\, q'}(E') = C_{\frac{2}{q},\, q'}(E).$$

Case 1: E is closed (thus $E_m = E$ for any $m \in \mathbb{N}$).
By Lemma 2.6 there exists a decreasing sequence $\{\varepsilon_{1,n}\}$ converging to 0, such that $\varepsilon_{1,1} < 1$, and

$$E \subset Q_1 = \bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_{1,n}}{2}} \subset \overline{Q}_1 \subset D_1,$$

where $K_n = B_n(x) \cap E$, $x \in E$. Also we have proven in Lemma 2.6 that the set $\bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_{1,n}}{2}}}$ is closed.
Again by Lemma 2.6 there exists a decreasing sequence $\{\varepsilon_{2,n}\}$ converging to 0, such that $\varepsilon_{2,n} \leq \varepsilon_{1,n}$ for all n and

$$E \subset Q_2 = \bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_{2,n}}{4}} \subset \overline{Q}_2 \subset D_2.$$

We note here that

$$\overline{Q}_2 \subset \bigcup_{n=1}^{\infty} \overline{K_n^{\frac{\varepsilon_{2,n}}{4}}} \subset \bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_{1,n}}{2}},$$

and since $\overline{K_n^{\frac{\varepsilon_{2,n}}{4}}}$ is closed we have

$$Q_2 \subset \overline{Q}_2 \subset Q_1.$$

By induction, we construct a decreasing sequence $\{\varepsilon_{j,n}\}$ converging to 0 with respect to n, such that $\forall n \in \mathbb{N} : \varepsilon_{j,n} \leq \varepsilon_{k,n}$ for all $j \geq k$,

$$E \subset Q_j = \bigcup_{n=1}^{\infty} K_n^{\frac{\varepsilon_{j,n}}{2^{j+1}}} \subset \overline{Q}_j \subset D_j,$$

and

$$Q_j \subset \overline{Q}_j \subset Q_{j-1}.$$

Now note that

$$E \subset Q_j \subset E^{\frac{1}{2^j}},$$

thus $E = \cap Q_j$. Finally,

$$C_{\frac{2}{q},\, q'}(E) \leq \lim C_{\frac{2}{q},\, q'}(Q_j) \leq \lim C_{\frac{2}{q},\, q'}(D_j) = C_{\frac{2}{q},\, q'}(E),$$

and the result follows in this case.
Case 2: E is not closed.
There exists a proper $\mathfrak{T}_q$-stratification of E, say $\{E_m\}$ and $E \sim^q E' = \bigcup_{i=1}^{\infty} E_i$. Also by the Case 1, we can assume without loss of generality that

$$E_{m+1} \setminus E_m \neq \emptyset \qquad \forall m \in \mathbb{N}.$$

Let us denote by Q_j^m the sets denoted by Q_j in the previous case if we replace E by E_m. Since there holds $C_{\frac{2}{q},\, q'}(\widetilde{E_m \setminus E_{m-1}}) \leq cC_{\frac{2}{q},\, q'}(E_m \setminus E_1)$, we can choose an open set D_m^1 such that $C_{\frac{2}{q},\, q'}(D_m^1) \leq \frac{c}{2^m}$. In view of Lemma (2.7) the set

$$Q_1 = \bigcup_{m=1}^{\infty} D_m^1 \cap Q_1^m$$

is open and

$$E' \subset Q_1 \subset \widetilde{Q}_1 \subset D_1.$$

Furthermore the set

$$\bigcup_{m=1}^{\infty} \widetilde{D_m^1 \cap Q_1^m}$$

is $\mathfrak{T}_q$-quasi closed. By Lemma 2.7 there exists an open set D_m^2 such that

$$D_m^2 \subset \widetilde{D}_m^2 \subset D_m^1.$$

By induction, we construct a sequence of open sets $\{D_m^j\}$ such that

$$D_m^j \subset \widetilde{D}_m^j \subset D_m^{j-1} \qquad C_{\frac{2}{q},\, q'}(D_m^j) \leq \frac{c}{2^m}.$$

Thus in view of Lemma 2.7 the set

$$Q_j = \bigcup_{m=1}^{\infty} D_m^j \cap Q_j^m$$

is open and the set

$$\bigcup_{m=1}^{\infty} \widetilde{D_m^j \cap Q_j^m}$$

is $\mathfrak{T}_q$-quasi closed. For any m we have

$$D_m^j \cap Q_j^m \subset \widetilde{D_m^j \cap Q_j^m} \subset \widetilde{D_m^j} \cap \widetilde{Q_j^m} \subset D_m^{j-1} \cap Q_{j-1}^m.$$

Thus

$$Q_j \subset \widetilde{Q_j} \subset \bigcup_{m=1}^{\infty} \widetilde{D_m^j \cap Q_j^m} \subset \bigcup_{m=1}^{\infty} D_m^{j-1} \cap Q_{j-1}^m \subset D_j.$$

Since the set $\bigcup_{m=1}^{\infty} \widetilde{D_m^j \cap Q_j^m}$ is $\mathfrak{T}_q$- quasi closed we have

$$Q_j \subset \widetilde{Q_j} \subset Q_{j-1}.$$

Finally

$$E' \subset Q_j \subset E'^{\frac{1}{2^j}},$$

thus $E' = \cap Q_j$. The result follows in this case since

$$C_{\frac{2}{q},\, q'}(E) \leq \lim C_{\frac{2}{q},\, q'}(Q_j) \leq \lim C_{\frac{2}{q},\, q'}(D_j) = C_{\frac{2}{q},\, q'}(E).$$

(II) The proof is same as in [23, Lemma 2.6 (b)] and we omit it. □

The next results are respectively proved in [23, Lemma 2.5] and [23, Lemma 2.7].

Proposition 2.9 *Let E be a bounded $\mathfrak{T}_q$-open set and let $\mathcal{D}$ be a cover of E consisting of $\mathfrak{T}_q$-open sets. Then, for every $\varepsilon > 0$ there exists an open set O_ε such that $C_{\frac{2}{q},\, q'}(O_\varepsilon) < \varepsilon$ and $E \setminus O_\varepsilon$ is covered by a finite subfamily of $\mathcal{D}$.*

Proposition 2.10 *Let Q be a $\mathfrak{T}_q$-open set. Then, for every $\xi \in Q$, there exists a $\mathfrak{T}_q$-open set O_ξ such that*

$$\xi \in Q_\xi \subset \widetilde{Q}_\xi \subset Q. \tag{2.38}$$

3 Lattice structure of $\mathcal{U}_+(Q)$

Consider the equation

$$\partial_t u - \Delta u + |u|^{q-1}u = 0, \quad \text{in } Q_\infty = \mathbb{R}^N \times (0, T], \text{ where } q \geq 1 + \frac{2}{N}. \tag{3.1}$$

A function $u \in L^q_{loc}(Q_T)$ is a subsolution (resp. supersolution) of the equation if $\partial_t u - \Delta u + |u|^{q-1}u \leq 0$ (resp. ≥ 0) holds in the sense of distributions.

If $u \in L^q_{loc}(Q_T)$ is a subsolution of the equation then by Kato's inequality $(\partial_t - \Delta)|u| + |u|^q \leq 0$ in the sense of distributions. Thus $|u|$ is a subsolution of the heat equation and consequently $u \in L^\infty_{loc}(Q_T)$. If $u \in L^q_{loc}(Q_T)$ is a solution then $u \in C^{2,1}(Q_T)$.

Proposition 3.1 *Let u be a non-negative function in $L^\infty_{loc}(Q_T)$.*
(i) If u is a subsolution of (3.1), there exists a minimal solution v dominating u, i.e. $u \leq v \leq U$ for any solution $U \geq u$.
(ii) If u is a supersolution of (3.1), there exists a maximal solution w dominated by u, i.e. $V \leq w \leq u$ for any solution $V \leq u$.
All the above inequalities hold almost everywhere .

Proof. (i) Let $\{J_\varepsilon\}$ be a filter of mollifiers in $\mathbb{R}^{N+1}$. If u is extended by zero outside of Q_T, then the function $u_\varepsilon = J_\varepsilon * u$ belong to $C^\infty(\mathbb{R}^{N+1})$, $\lim_{\varepsilon\to 0} u_\varepsilon = \widetilde{u} = u$ a.e. in $\mathbb{R}^{N+1}$ and $u_\varepsilon \to u$ in $L^q_{loc}(Q_T)$. We note that we can choose $\varepsilon > 0$ small enough such that the function u_ε is a subsolution in $B_R(0)\times(s,\infty)$ where $R > 0$ and $0 < s$. Let v_ε be the positive solution of

$$\begin{aligned} \partial_t v - \Delta v + |v|^{q-1}v = 0, &\qquad \text{in } B_R(0)\times(s,\infty),\\ v = u_\varepsilon, &\qquad \text{on } \partial B_R(0)\times(s,\infty),\\ v(.,s) = u_\varepsilon(.,s) &\qquad \text{in } B_R(0). \end{aligned} \tag{3.2}$$

In view of the proof of Lemma 2.4 and Remark 2.5 in [19] we can prove that $v_\varepsilon \geq u_\varepsilon$. Since v_ε is a subsolution of the heat equation, we have $v_\varepsilon \leq ||u_\varepsilon||_{L^\infty(B_R(0)\times(s,T])} \leq ||u||_{L^\infty(B_R(0)\times(s,T])}$. Thus there exists a decreasing sequence ε_j converging to 0 such that $v_{\varepsilon_j} \to v$ in $L^q(B_R(0)\times(s,T])$, $u \leq v \leq ||u||_{L^\infty(B_R(0)\times(s,T])}$; $0 < s < T < \infty$ and v is a positive solution of

$$\begin{aligned} \partial_t v - \Delta v + |v|^{q-1}v = 0, &\qquad \text{in } B_R(0)\times(s,T],\\ v = u, &\qquad \text{on } \partial B_R(0)\times(s,T],\\ v(.,s) = u(.,s) &\qquad \text{in } B_R(0). \end{aligned} \tag{3.3}$$

Let $\{R_j\}$ be an increasing sequence tending to infinity and s_j be a decreasing one converging to 0. Let v_j be the positive solution of the above problem with $R = R_j$ and $s = s_j$. Since $v_j \geq u$, we have by the maximum principle that $v_{j+1} \geq v_j$. Thus, by Keller-Osserman inequality and standard parabolic regularity results, there exists a subsequence, say $\{v_j\}$, such that $v_j \to v$ locally uniformly in Q_T. The results follows in this case by the construction of v.
(ii) Since $u \in L^q(B_R(0)\times(s,T])$ there exists a solution w of the problem

$$\begin{aligned} \partial_t w - \Delta w + |u|^q = 0, &\qquad \text{in } B_R(0)\times(s,T]\\ w = 0, &\qquad \text{on } \partial B_R(0)\times(s,T]\\ w(.,s) = 0 &\qquad \text{in } B_R(0). \end{aligned} \tag{3.4}$$

Hence $u+w$ is supersolution of the heat equation with boundary and initial data u. Consequently, $u + w \geq z$ where z is the solution of the heat equation with boundary and initial data u. Also, the function $z - w$ is a subsolution, thus there exists a solution $v \leq u$ of the problem (3.3) with boundary and initial data u. As before, let $\{R_j\}$ be an increasing sequence tending to infinity and s_j be a decreasing sequence tending to 0. Let v_j be the positive solution of the problem (3.3) with $R = R_j$ and $s = s_j$. Since $v_j \leq u$, we have by maximum principle that $v_{j+1} \leq v_j$. Thus by standard parabolic arguments, there exists a subsequence, say $\{v_j\}$, such that $v_j \to v$ locally uniformly in Q_∞. Again, the construction of v implies the result. □

Proposition 3.2 *Let u and v be nonnegative, locally bounded functions in Q_T.*
(i) If u and v are subsolutions (resp. supersolutions) then $\max(u,v)$ *is a subsolution (resp.* $\min(u,v)$ *is a supersolution).*
(ii) If u and v are supersolutions then $u+v$ is a supersolution.
(iii) If u is a subsolution and v is a supersolution then $(u-v)_+$ is a subsolution.

Proof. The first two statements are immediate consequence of the parabolic Kato's inequality. The third statement is verified in a similar way since

$$(\frac{d}{dt} - \Delta)(u-v)_+ \leq sign_+(u-v)(\frac{d}{dt} - \Delta)(u-v) \leq -sign_+(u-v)(u^q - v^q) \leq -(u-v)_+^q.$$

□

Notation 3.3 *Let u, v be nonnegative, locally bounded functions in Q_T.*
(a) If u is a subsolution, $[u]_\dagger$ denotes the smallest solution dominating u.
(b) If u is a supersolution, $[u]^\dagger$ denotes the largest solution dominated by u.
(c) If u, v are subsolutions then $u \vee v := [\max(u,v)]_\dagger$.
(d) If u, v are supersolutions then $u \wedge v := [\inf(u,v)]^\dagger$ and $u \oplus v = [u+v]^\dagger$.
(e) If u is a subsolution and v is a supersolution then $u \ominus v := [(u-v)_+]_\dagger$.

Proposition 3.4 *(i) Let $\{u_k\}$ be a sequence of positive, continuous subsolutions of (3.1). Then $U := \sup u_k$ is a subsolution. The statement remains valid if subsolution is replaced by supersolution and* sup *by* inf *.*
(ii) ([5]) Let $\mathcal{T}$ be a family of positive solutions of (3.1). Suppose that, for every u_1 and u_2 belonging to $\mathcal{T}$ there exists $v \in \mathcal{T}$ such that

$$\max(u_1,u_2) \leq v, \qquad \text{resp. } \min(u_1,u_2) \geq v.$$

Then there exists a monotone sequence $\{u_n\}$ in $\mathcal{T}$ such that

$$u_n \uparrow \sup \mathcal{T}, \qquad \text{resp. } u_n \downarrow \inf T.$$

Thus $\sup \mathcal{T}$ (resp. $\inf \mathcal{T}$) is a solution.

Proof. (i) Set $v_j = \max(u_1,u_2,...,u_j) = \max(\max(u_1,u_2), \max(\max(u_1,u_2),u_3),...,\max(\max(...),u_j))$. By proposition 3.2 v_j is a subsolution and $v_{j+1} \geq v_j$. Thus the positive solution $[v_j]_\dagger$ is increasing with respect to j. Also by Keller-Osserman inequality, we have that $[v_j]_\dagger \to \widetilde{v}$, where $\widetilde{v}$ is a positive solution. Thus $v_j \to v$ where v is a subsolution of (3.1). Now since $u_i \leq v$ for each $i \in \mathbb{N}$, we have that $U \leq v$. But $v_j \leq U$ for each $j \in \mathbb{N}$, which implies $v \leq U$. And thus $v = U$. The proof for "inf" is similar and we omit it.
(ii) The proof is similar as the one in [5]. Let $A = (x_n,t_n)$ be a countable dense subset of Q_T and let $u_{nm} \in \mathcal{T}$ satisfy the condition $\sup_m u_m(x_n,t_n) = w(x_n,t_n)$. Since $\mathcal{T}$ is closed with respect to $\vee$, there exists an increasing sequence of $v_n \in \mathcal{T}$ such that $v = \lim_{n\to\infty} v_n$, coincides with w on A. We claim that $v = w$ everywhere. Indeed, $v \leq u$. Suppose $u \in \mathcal{T}$. Then $u \leq w$ and therefore $u \leq v$ on A. Since A is everywhere dense and u, v are continuous, $u \leq v$ everywhere in Q_∞, which implies $u \geq w = \sup u$. □

As a consequence we have the following result which extends to equation (1.1) what Dynkin proved for (1.14) [5, Theorem 5.1].

Theorem 3.5 *The set $\mathcal{U}_+(Q_T)$ is a complete lattice stable for the laws $\oplus$ and $\ominus$.*

4 Partition of unity in Besov spaces

Lemma 4.1 *Let $U \subset \mathbb{R}^N$ be a $\mathfrak{T}_q$-open set and $z \in U$. Then there exists a function $f \in W^{\frac{2}{q},q'}(\mathbb{R}^N)$ with compact support in U such that $f(z) > 0$. In particular, there exists a bounded $\mathfrak{T}_q$-open set V such that $\overline{V} \subset U$.*

Proof. We suppose that z is not an interior point of U with respect to Euclidean topology, since otherwise the result is obvious. Since U is $\mathfrak{T}_q$-open we have that U^c is thin at z. Also by the assumption on z, we have that $z \in \overline{U^c} \setminus U$. By [2, p. 174], we can find an open set $W \supset U^c$, $z \in \overline{W} \setminus W$ and W is thin at z.
We recall that for a set E with positive $C_{\frac{2}{q},q'}$-capacity, $F^E := \mathcal{V}^{\mu_E} = G_{\frac{1}{q}} * (G_{\frac{1}{q}} * \mu_E)^{p-1}$ where μ_E is the capacitary measure on E. Then, by [2, Proposition 6.3.14], there exists $r > 0$ small enough such that

$$\mathcal{V}^{\mu}(z) < \frac{1}{2},$$

where μ is the capacitary measure of $B(z,r) \cap W$ and $\mathcal{V}^{\mu}$ the corresponding Besov potential (see [2, Theorems 2.2.7, 2.5.6]). By [2, Theorem 6.3.9], $\mathcal{V}^{\mu} \geq 1$ quasi everywhere (abr. q.a.e.) on $B(z,r) \cap W$, and by [2, Proposition 2.6.7] $\mathcal{V}^{\mu} \geq 1$ everywhere on $B(z,r) \cap W$. Thus

$$\mathcal{V}^{\mu}(z) < \frac{1}{2} < 1 \leq \mathcal{V}^{\mu}(x), \qquad \forall x \in B(z,r) \cap W.$$

Thus we can find $r_0 > 0$ small enough such that

$$\mathcal{V}^{\mu}(z) < \frac{1}{2} < 1 \leq \inf\{\mathcal{V}^{\mu}(x) : \ x \in B(z,r_0) \setminus U\}.$$

Now let $0 \leq H(t)$ be a smooth nondecreasing function such that $H(t) = t$ for $t \geq \frac{1}{4}$ and $H(t) = 0$ for $t \leq 0$. Also let $\eta \in C_0^{\infty}(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$, supp $\eta \subset B(z,r_0)$ and $\eta(z) = 1$. Then the function

$$f(z) = \eta H(1 - \mathcal{V}^{\mu}),$$

belongs to $W^{\frac{2}{q},q'}(\mathbb{R}^N)$. Since by definition $\mathcal{V}^{\mu}$ is lower semicontinuous, the set $\{1 - u \geq 0\}$ is closed. Hence the support of f is compact and

$$\text{supp} f \subset \text{supp}\eta \cap \{1 - u \geq 0\} \subset U.$$

□

Lemma 4.2 *Let U be a $\mathfrak{T}_q$-open set and $z \in U$. Then there exists a $\mathfrak{T}_q$-open set V, such that $z \in V \subset U$, and a function $\psi \in W^{\frac{2}{q},q'}(\mathbb{R}^N)$ such that $\psi = 1$ q.a.e. on V and $\psi = 0$ outside U.*

Proof. As before, we assume that z is not an interior point of U. Let $\mathcal{V}^{\mu}$ be the Besov potential of the previous lemma, with

$$\mathcal{V}^{\mu}(z) < \frac{1}{4}, \qquad \mathcal{V}^{\mu} = 1 \qquad \text{on } B(z,r_0) \setminus U.$$

By [2, Proposition 6.3.10] $\mathcal{V}^{\mu}$ is quasi continuous, that we can find a $\mathfrak{T}_q$-open set W which contains z such that

$$\mathcal{V}^{\mu}(x) \leq \frac{1}{4}, \quad \text{q.a.e. on } W.$$

Let $\eta \in C_0^{\infty}(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$, supp $\eta \subset B(z,r_0)$ and $\eta(x) = 1, \forall\, x \in B(z,\frac{r_0}{2})$. Set

$$f = 2\eta H\left(1 - H\left(\frac{1}{2} - \mathcal{V}^{\mu}(x)\right) - \mathcal{V}^{\mu}(x)\right).$$

Then $f \in W^{\frac{2}{q},q'}(\mathbb{R}^N)$, $0 \le f \le 1$ and $f = 0$ on $B(z,r_0) \setminus U$. Also, $f = 1$ on $B(z,\frac{r_0}{2}) \cap W$ and $f = 0$ outside of $B(z,r_0) \cap U$. $\square$

Lemma 4.3 *Let $\frac{2}{q} \le 1$, K be a compact set and U be a $\mathfrak{T}_q$-open set such that $K \subset U$. Also, let $\{U_j\}$ be a sequence of $\mathfrak{T}_q$-open subsets of U covering U up to a set of zero $C_{\frac{2}{q},\,q'}$-capacity Z. We assume that there exists a nonnegative $u \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp $u \subset K \subset U$. Then there exist $m(k) \in \mathbb{N}$ and nonnegative functions $u_{k,j} \in L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp $u_{k,j} \subset U_j$, such that*

$$\sum_{j=1}^{m(k)} u_{k,j} \le u \tag{4.1}$$

and

$$\lim_{k\to\infty} ||u - \sum_{j=1}^{m(k)} u_{k,j}||_{W^{\frac{2}{q},q'}(\mathbb{R}^N)} = 0.$$

Remark. If u changes sign, the conclusion of Lemma remains valid without inequality (4.1).

Proof. Without loss of generality we can assume that U and the $\cup_j U_j$ are bounded. For any $j \ge 0$, there exists open sets $G_{k,j}$ such that $C_{\frac{2}{q},\,q'}(G_{k,j}) \le 2^{-k-j}$, $Z \subset G_{k,0}$ and for $j \ge 1$, the sets $U_j \cup G_{k,j}$ are open. Also the sets

$$G_k = \bigcup_{j=0}^{\infty} G_{k,j}, \qquad \bigcup_{j=1}^{\infty} G_k \bigcup U_j$$

are open and $C_{\frac{2}{q},\,q'}(G_k) \to 0$ when $k \to \infty$.

Since G_k is open, its Besov potential F^{G_k} is larger or equal to 1 everywhere on G_k [2, Theorems 2.5.6, 2.6.7]). Also we have

$$||\mathcal{V}^{\mu_k}||^{q'}_{W^{\frac{2}{q},q'}(\mathbb{R}^N)} \le AC_{\frac{2}{q},\,q'}(G_k),$$

where A is a positive constant which depends only on $n,\ q$. Now consider a smooth nondecreasing function H such that $H(t) = 1$ for $t \ge 1$ and $H(t) = t$ for $t \le \frac{1}{2}$, then the function $\phi_k = H(\mathcal{V}^{\mu_k})$ belongs to $W^{\frac{2}{q},q'}(\mathbb{R}^N)$, satisfies $0 \le \phi_k \le 1$, $\phi_k = 1$ on G_k and there exists a constant $A'(n,q) > 0$ such that

$$||\phi_k||^{q'}_{W^{\frac{2}{q},q'}(\mathbb{R}^N)} \le A'C_{\frac{2}{q},\,q'}(G_k).$$

Set $\psi_k = 1 - \phi_k$. By Lebesgue's dominated theorem

$$||u - \psi_k u||^{q'}_{W^{\frac{2}{q},q'}(\mathbb{R}^N)} \to 0. \tag{4.2}$$

Thus it is enough to prove that

$$u\psi_k = \sum_{j=1}^{m(k)} u_{k,j}. \tag{4.3}$$

Fix $k \in \mathbb{N}$. Then there exist open balls $B_{k,j,i}$, for $i, j = 1, 2...$, such that

$$\overline{B}_{k,j,i} \subset U_j \bigcup G_k, \qquad \text{and} \bigcup_{j=1}^{\infty} G_k \bigcup U_j = \bigcup_{i,j=1}^{\infty} B_{k,j,i}.$$

Since K is compact, there exists $m(k) \in \mathbb{N}$ such that

$$K \subset \bigcup_{i,j=1}^{m(k)} B_{k,j,i}.$$

Now consider $w_{k,j,i} \in C_0^\infty(\mathbb{R}^N)$ such that

$$\{w_{k,j,i} > 0\} = B_{k,j,i}.$$

If we set

$$u_{k,j} = u\psi_k \frac{\sum_{i=1}^{m(k)} w_{k,j,i}}{\sum_{i,j=1}^{m(k)} w_{k,j,i}},$$

then $u_{k,j} \in L^\infty(\mathbb{R}^N)$, satisfies 1 and

$$\mathfrak{T}_q\text{-supp}u_{k,j} \subset (K \setminus G_k) \cap B_{k,j,i} \subset U_j.$$

$\square$

Remark. We conjecture that the result still holds if $\frac{2}{q} > 1$, but we have not been able to prove (4.2).

5 The regular set and its properties

Let $q > 1$, $T > 0$. If $Q_T = \mathbb{R}^N \times (0, T)$, we recall that $\mathcal{U}_+(Q_T)$ is the set of positive solutions u of

$$\partial_t u - \Delta u + u^q = 0 \qquad \text{in } Q_T. \tag{5.1}$$

If a function ζ is defined in $\mathbb{R}^N$. We denote by $\mathfrak{T}_q$-supp(ζ) the $\mathfrak{T}_q$-closure of the set where $|\zeta| > 0$. Let U be a Borel subset of $\mathbb{R}^N$ and χ_U be the characteristic function of U. We set

$$\mathbb{H}(\chi_U)(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \chi_U dy.$$

For any $\xi \in \mathbb{R}^N$ the following dichotomy occurs:

(i) either there exists a $\mathfrak{T}_q$-open bounded neighborhood $U = U_\xi$ of ξ such that

$$\int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\chi_U]^{2q'} dxdt < \infty, \tag{5.2}$$

where $q' = \frac{q}{q-1}$,
(ii) or for any $\mathfrak{T}_q$-open neighborhood U of ξ

$$\int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\chi_U]^{2q'} dxdt = \infty. \tag{5.3}$$

Definition 5.1 *The set of $\xi \in \mathbb{R}^N$ such that (i) occurs is $\mathfrak{T}_q$-open. It is denoted by $\mathcal{R}_q(u)$ and called the regular set of u. Its complement $\mathcal{S}_q(u) = \mathbb{R}^N \setminus \mathcal{R}_q(u)$ is $\mathfrak{T}_q$-closed and called the singular set of u.*

Proposition 5.2 *Let $\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support in a $\mathfrak{T}_q$-open bounded set U. Also let $u \in \mathcal{U}_+(Q_T)$ satisfy*

$$M_U = \int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\chi_U]^{2q'} dxdt < \infty.$$

Then there exists

$$l(\eta) := \lim_{t \to 0} \int_{\mathbb{R}^N} u(x,t) \mathbb{H}[\eta]_+^{2q'} dx. \tag{5.4}$$

Furthermore

$$|l(\eta)| \le C(M_U, q) \left(||\eta||^{2q'}_{W^{\frac{2}{q},q'}} + ||\eta||^{2q'}_{L^\infty(\mathbb{R}^N)} \right). \tag{5.5}$$

Proof. Put $h = \mathbb{H}[\eta]$ and $\phi(r) = r_+^{2q'}$. Since $|\eta| \le ||\eta||_{L^\infty} \chi_U$, there holds

$$\left| \int_0^T \int_{\mathbb{R}^N} u^q \phi(h) dxdt \right| \le ||\eta||^{2q'}_{L^\infty} \int_0^T \int_{\mathbb{R}^N} u^q \mathbb{H}[\chi_U]^{2q'} dxdt := ||\eta||^{2q'}_{L^\infty} M_U < \infty. \tag{5.6}$$

Moreover

$$\int_s^t \int_{\mathbb{R}^N} (-u(\partial_t \phi(h) + \Delta \phi(h))) + u^q \phi(h) dxd\tau = \int_{\mathbb{R}^N} u\phi(h)(.,s)dx - \int_{\mathbb{R}^N} u\phi(h)(.,t)dx. \tag{5.7}$$

But

$$\partial_t \phi(h) + \Delta \phi(h) = 2q' \phi(h) h_+^{-2} (2h_+ \partial_t h + (2q'-1)|\nabla h|^2).$$

By Hölder

$$\begin{aligned}
&\left| \int_s^t \int_{\mathbb{R}^N} u(\partial_t \phi(h) + \Delta \phi(h)) dxd\tau \right| \\
&\qquad \le \left(\int_s^t \int_{\mathbb{R}^N} u^q \phi(h) dxd\tau \right)^{\frac{1}{q}} \left(\int_s^t \int_{\mathbb{R}^N} \phi(h)^{-\frac{q'}{q}} |(\partial_t \phi(h) + \Delta \phi(h))|^{q'} dxd\tau \right)^{\frac{1}{q'}} \\
&\qquad \le 4q' \left(\int_s^t \int_{\mathbb{R}^N} u^q \phi(h) dxd\tau \right)^{\frac{1}{q}} \left(\int_s^t \int_{\mathbb{R}^N} (h_+ |\partial_t h| + |\nabla h|^2)^{q'} dxd\tau \right)^{\frac{1}{q'}}.
\end{aligned}$$

By standard regularity properties of the heat kernel

$$\int_s^t \int_{\mathbb{R}^N} |\partial_t h|^{q'} dxd\tau \le \int_0^T \int_{\mathbb{R}^N} |\partial_t h|^{q'} dxd\tau \le ||\eta||^{q'}_{W^{\frac{2}{q},q'}},$$

and by Gagliardo-Nirenberg inequality and the maximum principle

$$\int_s^t \int_{\mathbb{R}^N} |\nabla h|^{2q'} dxd\tau \le \int_0^T \int_{\mathbb{R}^N} |\nabla h|^{2q'} dxd\tau \le C ||\eta||^{q'}_{L^\infty} ||\Delta h||^{q'}_{L^{q'}} = C ||\eta||^{q'}_{L^\infty} ||\partial_t h||^{q'}_{L^{q'}}.$$

Therefore,

$$\left|\int_s^t\int_{\mathbb{R}^N} u(\partial_t\phi(h)+\Delta\phi(h))dxd\tau\right| \leq C\left(\int_s^t\int_{\mathbb{R}^N} u^q\phi(h)dxd\tau\right)^{\frac{1}{q}}||\eta||_{L^\infty}||\eta||_{W^{\frac{2}{q},q'}}. \tag{5.8}$$

This implies that the left-hand side of (5.7) tends to 0 when $s,t\to 0$, thus there exists

$$l(\eta) := \lim_{s\to 0}\int_{\mathbb{R}^N} u\phi(h)(x,s)dx.$$

From (5.7) it follows

$$\int_0^T\int_{\mathbb{R}^N}(-u(\partial_t\phi(h)+\Delta\phi(h)))+u^q\phi(h)dxd\tau+\int_{\mathbb{R}^N} u\phi(h)(.,T)dx = l(\eta). \tag{5.9}$$

Since $|u\phi(h)(.,T)| \leq C(T)||\eta||^{2q'}_{L^\infty}$, we derive

$$|l(\eta)| \leq C_1||\eta||^{2q'}_{L^\infty}+C||\eta||^{q'}_{L^\infty}||\eta||^{q'}_{W^{\frac{2}{q},q'}} \leq C\left(||\eta||_{L^\infty}+||\eta||_{W^{\frac{2}{q},q'}}\right)^{2q'}. \tag{5.10}$$

Proposition 5.3 *Let the assumptions of Lemma 5.2 be satisfied. Then*

$$\lim_{t\to 0}\int_U u(x,t)\eta_+^{2q'}(x)dx = l(\eta). \tag{5.11}$$

Proof. Using (5.6) with h replaced by $h_s(x,t) := \mathbb{H}[\eta](x,t-s)$, we get

$$\int_s^T\int_{\mathbb{R}^N}(-u(\partial_t\phi(h_s)+\Delta\phi(h_s)))+u^q\phi(h_s)dxd\tau+\int_{\mathbb{R}^N} u\phi(h_s)(.,T)dx = \int_{\mathbb{R}^N} u\phi(h_s)(.,s)dx. \tag{5.12}$$

When $s\to 0$

$$\int_{\mathbb{R}^N} u\phi(h_s)(.,T)dx \to \int_{\mathbb{R}^N} u\phi(h)(.,T)dx,$$

and

$$\int_s^T\int_{\mathbb{R}^N} u^q\phi(h_s)dxd\tau \to \int_0^T\int_{\mathbb{R}^N} u^q\phi(h)dxd\tau,$$

by the dominated convergence theorem. Furthermore,

$$\begin{aligned}&\left|\int_0^{T-s}\int_{\mathbb{R}^N}(u(x,t+s)-u(x,t))(\partial_t\phi(h)+\Delta\phi(h))dxdt\right|\\ &\qquad\leq C\left(\int_0^{T-s}\int_{\mathbb{R}^N}|u(x,t+s)-u(x,t)|^q h_+^{2q'}dxdt\right)^{\frac{1}{q}}||\eta||^{q'}_{L^\infty}||\eta||^{q'}_{W^{\frac{2}{q},q'}},\end{aligned}$$

which tends to zero with s. Finally,

$$\lim_{s\to 0}\int_{T-s}^T\int_{\mathbb{R}^N} u^q\phi(h)dxd\tau = 0.$$

Subtracting (5.7) to (5.12), we derive

$$\lim_{s\to 0}\int_{\mathbb{R}^N} u(.,s)(\phi(h)(.,s)-\phi(\eta))dx = 0,$$

which implies the claim. $\square$

The next statement obtained by contradiction with the use of Lemma 5.2 and Lemma 5.3 will be very useful in the sequel.

Proposition 5.4 *Assume that U is a bounded $\mathfrak{T}_q$-open set and*

$$\lim_{t\to 0}\int_U u(x,t)\eta^{2q'}(x)dx = \infty, \tag{5.13}$$

for some $0 \le \eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support in U, then

$$\int_0^T\int_{\mathbb{R}^N} u^q\mathbb{H}[\eta]^{2q'}dxdt = \infty. \tag{5.14}$$

Proposition 5.5 *Let $\xi \in \mathcal{S}_q(u)$. Then for any $\mathfrak{T}_q$-open set G which contains ξ, there holds*

$$\lim_{t\to 0}\int_G u(x,t)dx = \infty. \tag{5.15}$$

Proof. If $\xi \in \mathcal{S}_q(u)$ and if G is $\mathfrak{T}_q$-open and contains ξ, then by Lemma 4.2 there exist $\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and a $\mathfrak{T}_q$-open set $D \subset G$ such that $\eta = 1$ on D, $\eta = 0$ outside of G and $0 \le \eta \le 1$. Thus

$$\infty = \int_0^T\int_{\mathbb{R}^N} u^q\mathbb{H}[\chi_D]^{2q'}dxdt \le \int_0^T\int_{\mathbb{R}^N} u^q\mathbb{H}.[\eta]^{2q'}dxdt,$$

Therefore

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u\mathbb{H}[\eta]^{2q'}dx = \infty,$$

which implies

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u\eta^{2q'}dx = \infty,$$

and the result follows by the properties of η. $\square$

5.1 Moderate solutions

We first recall some classical results concerning initial value problem with initial measure data. A solution u of (3.1) is called *moderate* if $u \in L^q(K)$ for any compact $K \subset \overline{Q}_\infty$. Then there exists a unique Radon measure μ such that

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N}\zeta(x)d\mu \qquad \forall \zeta \in C_0^\infty(\mathbb{R}^N). \tag{5.16}$$

Equivalently

$$-\int\int_{Q_\infty} u(\phi_t + \Delta\phi)dxdt + \int\int_{Q_\infty} |u|^{q-1}u\phi dxdt = \int_{\mathbb{R}^N}\phi(x,0)d\mu,$$

for all $\phi \in C^{1,1;1}(\overline{Q}_\infty)$, with compact support.

The above measure has the property that it vanishes on Borel sets with $C_{\frac{2}{q},\,q'}$-capacity zero. There exists an sequence $\{\mu_n\} \subset W^{-\frac{2}{q},q}(\mathbb{R}^N)$ of Radon measures such that $\mu_n \rightharpoonup \mu$ in the weak* topology. If we assume that u is a positive moderate solution, or equivalently that the initial measure μ is positive, then the previous sequence can be constructed as being increasing and particularly $\{\mu_n\} \subset W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$, where $\mathfrak{M}_+^b(\mathbb{R}^N)$ is the set of all positive bounded Radon measures in $\mathbb{R}^N$.

If $\nu \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$, then we have for some constant $C > 0$ independent on ν(see Lemma 3.2-[22])

$$C^{-1}||\nu||_{W^{-\frac{2}{q},q}(\mathbb{R}^N)} \leq ||\mathbb{H}[\nu]||_{L^q(Q_T)} \leq C||\nu||_{W^{-\frac{2}{q},q}(\mathbb{R}^N)}, \tag{5.17}$$

where we recall that $\mathbb{H}[\nu]$ denotes the heat potential of ν in Q.

Lemma 5.6 *Let u be a moderate positive solution with initial data μ. Then for any $T > 0$ and bounded $\mathfrak{T}_q$-open set we have*

$$\int_0^T\int_{\mathbb{R}^N} u^q(t,x)\mathbb{H}^{2q'}[\chi_O]dxdt < \infty.$$

Proof. Let $0 \leq \eta \in C_0^\infty(\mathbb{R}^N)$ and $\eta = 1$ on O and $s < T$. We define here $h = \mathbb{H}[\eta](x,t)$, $h_s = \mathbb{H}[\eta](x,t-s)$ and $\phi(r) = |r|^{2q'}$. Then we have

$$\int_s^T\int_{\mathbb{R}^N} u(x,t)\left(\partial_t\phi(h_s) + \Delta\phi(h_s)\right) + |u|^q\phi(h_s)dxdt + \int_{\mathbb{R}^N} u\phi(h_s)(.,T)dx = \int_{\mathbb{R}^N} u(x,s)\phi(\eta)dx.$$

In view of Proposition 5.2, (5.8) and Hölder's inequality, there exists a constant $c = c(q,N)$ such that

$$\int_s^T\int_{\mathbb{R}^N}|u|^q\phi(h_s)dxdt + \int_{\mathbb{R}^N} u\phi(h_s)(.,T)dx \leq c\left(\int_{\mathbb{R}^N} u(x,s)\phi(\eta)dx + ||\eta||_{L^\infty}^{2q'}||\eta||_{W^{\frac{2}{q},q'}}^{2q'}\right).$$

Using Fatou's lemma and the fact that, for any bounded Borel set E

$$\limsup_{s\to 0}\int_E u(x,s)dx < \infty,$$

we conclude the proof. $\square$

Theorem 5.7 *Let u be a positive moderate solution with μ as initial data, then*
(i) μ is regular relative to the $\mathfrak{T}_q$-topology.
(ii) For each quasi continuous function $\phi \in L^\infty(\mathbb{R}^N)$ with bounded $\mathfrak{T}_q$-support in $\mathbb{R}^N$, we have

$$\lim_{t\to 0}\int_{\mathbb{R}^N} u(x,t)\phi(x)dx = \int_{\mathbb{R}^N}\phi(x)d\mu.$$

Proof. The proof is similar to the one given [23].
(i) Every Radon measure on $\mathbb{R}^N$ is regular in the usual Euclidean topology, i.e.

$$\begin{aligned}\mu(E) &= \inf\{\mu(D):\ E\subset D,\ D \text{ open}\}\\ &= \inf\{\mu(K):\ K\subset E,\ K \text{ compact}\},\end{aligned}$$

for any Borel set E. But if D is open and contains E, it is $\mathfrak{T}_q$-open, hence

$$\mu(E)\le \inf\{\mu(D):\ E\subset D,\ D\ \mathfrak{T}_q\text{-open}\}\le \inf\{\mu(D):\ E\subset D,\ D \text{ open}\}=\mu(E),$$

and the result follows.
(ii) Since the measure $\mu_t = u(t,x)dx \rightharpoonup \mu$ in the weak* topology we have

$$\limsup_{t\to 0}\mu_t(E)\le \mu(E), \qquad \liminf_{t\to 0}\mu_t(A)\ge\mu(A),$$

for any compact set E, respectively, open set A. This extends to any bounded $\mathfrak{T}_q$-closed set E (resp. $\mathfrak{T}_q$-open set A).
Indeed, let E be a $\mathfrak{T}_q$-closed set and $\{K_n\}$ be an increasing sequence of closed sets such that $C_{\frac{2}{q},\,q'}(E\setminus K_n)\to 0$. Then for any $m\in\mathbb{N}$ and any open set $E\subset O$ we have

$$\limsup_{t\to 0}\mu_t(E)\le \limsup_{t\to 0}\mu_t(K_m)+\limsup_{t\to 0}\mu_t(E\setminus K_m)\le \mu(O)+\limsup_{t\to 0}\mu_t(E\setminus K_m).$$

Now we assert that

$$\lim_{m\to\infty}\limsup_{t\to 0}\mu_t(E\setminus K_m)=0.$$

We will prove it by contradiction. We assume that $\lim_{m\to\infty}\limsup_{t\to 0}\mu_t(E\setminus K_m)=\varepsilon>0$.
Let $\{t_n\}$ be a decreasing sequence tending to 0 and $\lim_{n\to\infty}\mu_{t_n}(E\backslash K_m)=\limsup_{t\to 0}\mu_t(E\backslash K_m)$. Then there exists subsequence of positive solutions $\{u_k^m\}_{k=1}^\infty$ with initial data $\mu_{t_{n_k}}\chi_{E\backslash K_m}$ such that $u_k^m\to u^m$ for any $m\in\mathbb{N}$. Since u is a moderate solution and $u_k^m\le u$, u^m is a moderate solution too. Also by construction, the sequence $\{u^m\}$ is nonincreasing and $u_m\le U_{E\backslash K_m}$. By proposition 5.17 we have $U_{E\backslash K_m}\to 0$ which implies $u_m\to 0$ and

$$\lim_{m\to\infty}\lim_{k\to\infty}\mu_{t_{n_k}}(E\setminus K_m)=0.$$

The proof follows in the case where E is $\mathfrak{T}_q$-closed. The proof is similar in the other case.
If A is $\mathfrak{T}_q$-open and

$$\mu(A)=\mu(\widetilde{A}),$$

then

$$\lim_{t\to 0}\mu_t(A)=\mu(A).$$

Without loss of generality we may assume that $\phi\geq 0$ (since otherwise we set $\phi=\phi^+-\phi^-$) and $\phi\leq 1$. Given $k\in\mathbb{N}$ and $m=0,...,2^k-1$ choose a number $a_{m,k}$ in the interval $(m2^{-k},(m+1)2^{-k})$ such that $\mu(\phi^{-1}(\{a_{m,k}\}))=0$. Put

$$A_{m,k}=\phi^{-1}((a_{m,k},(a_{m+1,k}]),\;\; m=1,...,2^k-1, \qquad A_{0,k}=\phi^{-1}((a_{0,k},(a_{1,k}]),$$

then we note that since ϕ has compact support the above sets are bounded and

$$\lim_{t\to 0}\mu_t(A_{m,k})=\mu(A_{m,k}),\forall m\geq 0,\; k\in\mathbb{N}. \tag{5.18}$$

Consider the step function $\phi_k=\sum_{\mu=0}^{2^k-1}m2^{-k}\chi_{A_{m,k}}$, then $\phi_k\uparrow\phi$ uniformly, and by (5.18),

$$\lim_{t\to 0}\int_{\mathbb{R}^N}u(x,t)\phi_k dx=\int_{\mathbb{R}^N}\phi_k d\mu, \qquad \forall\zeta\in C_0^\infty(\mathbb{R}^N).$$

This completes the proof of (ii). □

5.2 Vanishing properties

Definition 5.8 *A continuous function $u\in\mathcal{U}_+(Q_T)$ vanishes on a $\mathfrak{T}_q$-open subset $G\subset\mathbb{R}^N$, if for any $\eta\in W^{\frac{2}{q},q'}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp$(\eta)\subset^q G$, there holds*

$$\lim_{t\to 0}\int_G u(x,t)\eta_+^{2q'}(x)dxdt=0. \tag{5.19}$$

When this is case we write $u\approx_G 0$. We denote by $\mathcal{U}_G(Q_T)$ the set of $u\in\mathcal{U}_+(Q_T)$ which vanish on G.

We have the following simple result.

Proposition 5.9 *Let A be a $\mathfrak{T}_q$-open subset of $\mathbb{R}^N$ and $u_1,\; u_2\in\mathcal{U}_+(Q_T)$.*
If $u_2\approx_A 0$ and $u_1\leq u_2$ then $u_1\approx_A 0$.

Proposition 5.10 *Let G,G' be $\mathfrak{T}_q$-open sets such that $G\sim^q G'$. If $u\in\mathcal{U}_G(Q_T)$ then $u\in\mathcal{U}_{G'}(Q_T)$*

Proof. If $\eta\in W^{\frac{2}{q},q'}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp$(\zeta)\subset^q G$, then $\mathfrak{T}_q$-supp$(\zeta)\subset^q G'$. Since $|G\setminus G'|=|G'\setminus G|=0$ the result follows. □

If G is an open subset, this notion coincides with the usual definition of vanishing, since we can take test function $\eta\in C_0^\infty(G)$. In that case $u\in C(Q_T\cup\{G\times\{0\}\})$.

Lemma 5.11 *Assume $u\in\mathcal{U}_G(Q_T)$. Then for any $\eta\in W^{\frac{2}{q},q'}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp$(\eta)\subset^q G$, there holds*

$$\int_0^T\int_{\mathbb{R}^N}u^q\mathbb{H}[\eta]_+^{2q'}dxdt+\int_{\mathbb{R}^N}u(x,T)\mathbb{H}[\eta]_+^{2q'}(x,T)dx\leq C_1||\eta||_{L^\infty}^{q'}||\eta||_{W^{\frac{2}{q},q'}}^{q'}. \tag{5.20}$$

Proof. If $u \in \mathcal{U}_G(Q_T)$ and $\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-supp$(\eta) \subset^q G$, there holds, with $h = \mathbb{H}[\eta]$ and $\phi(r) = r_+^{2q'}$.

$$\int_0^T \int_{\mathbb{R}^N} (-u(\partial_t\phi(h) + \Delta\phi(h))) + u^q\phi(h)dxd\tau + \int_{\mathbb{R}^N} u\phi(h)(.,T)dx = 0. \tag{5.21}$$

Therefore (5.20) follows from (5.8). $\square$

Lemma 5.12 *Let $G \subset \mathbb{R}^N$ be a $\mathfrak{T}_q$-open set. Then there exists a nondecreasing sequence $\{u_n\} \subset \mathcal{U}_G(Q_T)$ which converges to* $\sup\mathcal{U}_G(Q_T)$. *Furthermore* $\sup\mathcal{U}_G(Q_T) \in \mathcal{U}_G(Q_T)$.

Proof. If u_1 and u_2 belongs to $\mathcal{U}_G(Q_T)$, then $u_1 + u_2$ is a supersolution and it satisfies (5.19). Therefore $u_1 \vee u_2$ is a solution which is smaller than $u_1 + u_2$, thus $u_1 \vee u_2 \in \mathcal{U}_G(Q_T)$. By Proposition 3.4 there exists a increasing sequence $\{u_n\} \subset \mathcal{U}_G(Q_T)$ which converges to $u := \sup\mathcal{U}_G(Q_T)$. By (5.21),

$$\int_0^T \int_{\mathbb{R}^N} (-u_n(\partial_t\phi(h) + \Delta\phi(h))) + u_n^q\phi(h)dxd\tau + \int_{\mathbb{R}^N} u_n\phi(h)(.,T)dx = 0. \tag{5.22}$$

Now, $u_n^q\phi(h) \uparrow u^q\phi(h)$ in $L^1(Q_T)$ and $u_n\phi(h)(.,T) \uparrow u\phi(h)(.,T)$ in $L^1(\mathbb{R}^N)$. If E is any Borel subset of Q_T, there holds by Hölder's inequality, as in (5.8)

$$\left| \int_0^T \int_E u_n(\partial_t\phi(h) + \Delta\phi(h))dxd\tau \right| \leq C \left(\int_0^T \int_E u_n^q\phi(h)dxd\tau \right)^{\frac{1}{q}} ||\eta||_{L^\infty} ||\eta||_{W^{\frac{2}{q},q'}}. \tag{5.23}$$

The right-hand side tends to zero when $|E| \to 0$, thus by Vitali's convergence theorem, we derive

$$\int_0^T \int_{\mathbb{R}^N} (-u(\partial_t\phi(h) + \Delta\phi(h))) + u^q\phi(h)dxd\tau + \int_{\mathbb{R}^N} u\phi(h)(.,T)dx = 0, \tag{5.24}$$

from (5.22). Thus $u \in \mathcal{U}_G(Q_T)$. $\square$

Definition 5.13 *(a) Let $u \in \mathcal{U}_+(Q_T)$ and let A denote the union of all $\mathfrak{T}_q$-open sets on which u vanishes. Then A^c is called the fine initial support of u, to be denoted by $\mathfrak{T}_q$-supp(u).*
(b) Let F be a Borel subset of $\mathbb{R}^N$. We denote by U_F the maximal element of $\mathcal{U}_{\widetilde{F}^c}(Q_T)$.

5.3 Maximal solutions

Definition 5.14 *Let $\mathfrak{M}_+^b(\mathbb{R}^N)$ be the set of all positive bounded Radon measures in $\mathbb{R}^N$. Also let $u_\mu \in \mathcal{U}_+(Q_T)$ be the moderate solution with initial data μ.*
For any Borel set $E \subset \mathbb{R}^N$ of positive $C_{\frac{2}{q},\,q'}$-capacity put

$$\mathcal{V}_{mod}(E) = \{u_\mu : \ \mu \in W^{-\frac{2}{q},q'}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N), \ \mu(E^c) = 0\}.$$

$$V_E = \sup V_{mod}(E).$$

The following result due to Marcus and Véron [22] shows that the maximal solution which vanishes on an open set is indeed σ-moderate. This is obtained by proving a capacitary quasi-representation of the solution via a Wiener type test.

Proposition 5.15 *Let F be a closed subset of $\mathbb{R}^N$ and $q \geq 1+\frac{2}{N}$. Then there exist two positive constants $C_1,\ C_2>0$, depending only on N and q such that*

$$\begin{aligned}C_1 t^{-\frac{1}{q-1}}\sum_{k=0}^{\infty}(k+1)^{\frac{N}{2}}\,e^{-\frac{k}{4}}C_{\frac{2}{q},\,q'}\left(\frac{F\cap F_k(x,t)}{\sqrt{(k+1)t}}\right) &\leq V_F(x,t)\leq U_F(x,t)\\ \leq C_2 t^{-\frac{1}{q-1}}\sum_{k=0}^{\infty}(k+1)^{\frac{N}{2}}\,e^{-\frac{k}{4}}C_{\frac{2}{q},\,q'}&\left(\frac{F\cap F_k(x,t)}{\sqrt{(k+1)t}}\right)\quad \forall (x,t)\in Q,\end{aligned}\tag{5.25}$$

where $F_k(x,t)=\{y\in\mathbb{R}^N:\ \sqrt{kt}\leq|x-y|\leq\sqrt{(k+1)t}\}$. As a consequence $U_F=V_F$.

Remark. We recall that the main argument for proving uniqueness is the fact that

$$U_F\leq\frac{C_2}{C_1}V_F\qquad\text{in } Q.\tag{5.26}$$

This argument introduced in [17] for elliptic equations has been extended to parabolic equations in [19], [22].

Definition 5.16 *Let F be a Borel subset of $\mathbb{R}^N$. We denote by U_F the maximal element of $\mathcal{U}_{\widetilde{F}^c}(Q_T)$.*

Proposition 5.17 *If $\{A_n\}$ is a sequence of Borel sets such that $C_{\frac{2}{q},\,q'}(A_n)\to 0$, then $U_{A_n}\to 0$.*

Proof. Let O_n be an open set such that $A_n\subset O_n$ and $C_{\frac{2}{q},\,q'}(O_n)\leq C_{\frac{2}{q},\,q'}(A_n)+\frac{1}{n}$. Now since O_n is open, $C_{\frac{2}{q},\,q'}$ is an outer measure, by (2.36) and (iv)-Proposition 2.3, we have

$$C_{\frac{2}{q},\,q'}(\overline{O}_n)=C_{\frac{2}{q},\,q'}\left((\overline{O}_n\cap b_q(O_n))\bigcup(\overline{O}_n\cap e_q(\overline{O}_n))\right)\leq C_{\frac{2}{q},\,q'}(\widetilde{O}_n)\leq cC_{\frac{2}{q},\,q'}(O_n).$$

Thus $C_{\frac{2}{q},\,q'}(\overline{O}_n)\to 0$. The result follows by

$$U_{A_n}\leq U_{\overline{O}_n}$$

and by (5.25). □

Corollary 5.18 *Let E be a Borel set such that $C_{\frac{2}{q},\,q'}(E)=0$. If $u\in\mathcal{U}_{\widetilde{E}^c}(Q_T)$ then $u=0$. In particular $U_E\equiv 0$.*

Proposition 5.19 *Let $E,\ F$ be Borel sets.*
(i) If $E,\ F$ are $\mathfrak{T}_q$-closed, then $U_E\wedge U_F=U_{E\cap F}$.
(ii) If $E,\ F$ are $\mathfrak{T}_q$-closed, then

$$\begin{aligned}U_E<U_F\quad&\Leftrightarrow\quad[E\subset^q F\text{ and } C_{\frac{2}{q},\,q'}(F\setminus E)>0],\\ U_E=U_F\quad&\Leftrightarrow\quad E\sim^q F.\end{aligned}\tag{5.27}$$

(iii) If F_n is a decreasing sequence of $\mathfrak{T}_q$-closed sets, then

$$\lim_{n\to\infty} U_{F_n} = U_F \text{ where } F = \cap F_n.$$

(iv) Let A be a $\mathfrak{T}_q$-open set and $u \in \mathcal{U}_+(Q_T)$. Suppose that u vanishes $\mathfrak{T}_q$-locally in A, i.e. for every point $\sigma \in A$ there exists a $\mathfrak{T}_q$-open set A_σ such that

$$\sigma \in A \subset A, \quad u \approx_{A_\sigma} 0.$$

Then u vanishes on A. In particular any $u \in \mathcal{U}_+(Q_T)$ vanishes on the complement of $\mathfrak{T}_q$-supp (u).

Proof. The proof is similar to the one in [23] dealing with elliptic equations.
(i) $U_E \wedge U_F$ is the largest solution under $\inf(U_E, U_F)$ and therefore, by definition, it is the largest solution which vanishes outside $E \cap F$.
(ii) By (5.25) U_E and U_F satisfies the same capacitary quasi-representation up to universal constants. By the Remark after Proposition 5.15 ,

$$E \sim^q F \Rightarrow \frac{C_1}{C_2} U_E \leq U_F \leq \frac{C_2}{C_1} U_E \Rightarrow U_E = U_F.$$

The proof of

$$E \subset^q F \Rightarrow U_E \leq U_F.$$

follows from Proposition 5.15 and the fact that $U_E = V_E$ and $U_F = V_F$ and $V_E \leq V_F$. In addition,

$$C_{\frac{2}{q}, q'}(F \setminus E) > 0 \Rightarrow U_E \neq U_F.$$

Indeed, if K is a compact subset of $F \setminus E$ of positive capacity, then $U_K > 0$ and $U_K \leq U_F$ but $U_K \not\leq U_E$. Therefore $U_E = U_F$ implies $E \sim^q F$ and $U_E \leq U_F$ implies $E \subset^q F$.
(iii) If $V := \lim_{n\to\infty} U_{F_n}$ then $U_F \leq V$. But $\mathfrak{T}_q$-supp $(V) \subset F_n$ for each $n \in \mathbb{N}$ and consequently $V \leq U_F$.
(iv) First assume that A is a countable union of $\mathfrak{T}_q$-open sets $\{A_n\}$ such that $u \approx_{A_n} 0$ for each n. Then u vanishes on $\cup_{i=1}^k A_k$ for each k. Therefore we can assume that the sequence A_k is increasing. Put $F_n = A_n^c$. Then $u \subset U_{F_n}$ and by (iii), $U_{F_n} \downarrow U_F$ where $F = A^c$. Thus $u \leq U_F$, i.e.,which is equivalent to $u \approx_A 0$.

We turn to the general case. It is known that the $\mathfrak{T}_q$-fine topology possesses the *quasi-Lindelöf property* (see [2, Sec. 6.5.11]) as any topology associated to a Bessel capacity $C_{\alpha,p}$. Therefore A is covered, up to a set of capacity zero, by a countable subcover of $\{A_\sigma : \sigma \in A\}$. Therefore the previous argument implies that $u \approx_A 0$. □

Proposition 5.20 *(i) Let E be a $\mathfrak{T}_q$-closed set. Then*

$$\begin{aligned} U_E &= \inf\{U_D : E \subset D, \ D \text{ open}\} \\ &= \sup\{U_K : K \subset E, \ K \text{ closed}\}. \end{aligned} \tag{5.28}$$

(ii) If E, F are two Borel sets then

$$U_E = U_{F\cap E} \oplus U_{E\setminus F}.$$

(iii) Let $E,\ F_n,\ n=1,2,...$ be Borel sets and let u be a positive solution of (3.1). If either $C_{\frac{2}{q},\,q'}(E\bigtriangleup F_n)\to 0$ or $\widetilde{F}_n\downarrow\widetilde{E}$ then

$$U_{F_n}\to U_E.$$

Proof. (i) Let $\{Q_j\}$ be the decreasing sequence of open sets of Lemma 2.8-(I) such that $\cap Q_j=\cap\widetilde{Q}_j=E'\sim^q E$. Thus by Proposition 5.19 (iii) we have that $U_{Q_j}\to U_E$, this implies the first equality in (i).

Let $\{F_n\}$ be a nondecreasing sequence of closed subsets of E such that $C_{\frac{2}{q},\,q'}(E\setminus F_n)\to 0$. Let D_1 and D_2 be open sets such that $F_n\subset D_1$ and $E\setminus F_n\subset D_2$. Also set $D_3=(\widetilde{D}_1\cup\widetilde{D}_2)^c$. Let $u^{(i)}_\beta$ be the positive solution of

$$\begin{array}{rcll}\partial_t u-\Delta u+u^q&=&0&\text{in }\ \mathbb{R}^N\times(\beta,T]\\ u(.,\beta)&=&\chi_{\widetilde{D}_i}U_E(.)&\text{on }\ \mathbb{R}^N,\end{array}\tag{5.29}$$

where $0<\beta<T$. For any $(x,t)\in\mathbb{R}^N\times(\beta,T]$ we have

$$U_E\le u^{(1)}_\beta+u^{(2)}_\beta+u^{(3)}_\beta.$$

Letting $\beta\to 0$ (taking an subsequence if it is necessary) we have $u^{(i)}_\beta\to u^{(i)}$ and

$$U_E\le u^{(1)}+u^{(2)}+u^{(3)}\qquad\text{in }\ Q_T,$$

But $u^{(i)}\le U_{D_i}$ thus

$$U_E\le U_{D_1}+U_{D_2}+u^{(3)}.$$

Now $u^{(3)}\le U_{D_3}$ and $u^{(3)}\le U_E$ thus by Proposition 5.20-(i) $u^{(3)}\le U_{D_3\cap E}$. But $D_1\cup D_2$ is an open set and thus $C_{\frac{2}{q},\,q'}(D_3\cap E)=0$, which implies by Corollary 5.18 that $u^{(3)}=0$. Finally we have that

$$U_E\le U_{D_1}+U_{D_2}.$$

Since D_i is arbitrary, we have by the first assertion of this Proposition

$$U_E\le U_{F_n}+U_{E\setminus F_n}.\tag{5.30}$$

But $C_{\frac{2}{q},\,q'}(E\setminus F_n)\to 0$, thus by Proposition 5.17, we have

$$U_E\le\lim_{n\to\infty}U_{F_n}\Rightarrow U_E=\lim_{n\to\infty}U_{F_n},$$

since $U_{F_n}\le U_E$ for any $n\in\mathbb{N}$.
(ii) By similar argument as in the proof of (5.30) we can prove that

$$U_E\le U_{F\cap E}+U_{E\setminus F}\Rightarrow U_E\le U_{F\cap E}\oplus U_{E\setminus F}.$$

On the other hand both $U_{F\cap E}$ and $U_{E\setminus F}$ vanish outside of $\widetilde{E}$. Consequently $U_{F\cap E}\oplus U_{E\setminus F}$ vanishes outside $\widetilde{E}$ so that

$$U_E\ge U_{F\cap E}\oplus U_{E\setminus F},$$

and the result follows in this statement.
(iii) The previous statement implies,

$$U_E \leq U_{F_n\cap E} + U_{E\setminus F_n}, \qquad U_{F_n} \leq U_{F_n\cap E} + U_{F_n\setminus E}. \tag{5.31}$$

If $C_{\frac{2}{q},\, q'}(E \bigtriangleup F_n) \to 0$ then Proposition 5.17 implies $U_{E\bigtriangleup F_n} \to 0$. And the result follows in this case by (5.31).
If $\widetilde{F}_n \downarrow \widetilde{E}$ the result follows in this case by Proposition 5.19(iii). $\square$

This implies the following extension of Proposition 5.15 to merely $\mathfrak{T}_q$-closed sets.

Proposition 5.21 *If E is a $\mathfrak{T}_q$-closed set, then V_E and U_E satisfy the capacitary estimates (5.15). Thus $U_E = V_E$ and the maximal solution U_E is σ-moderate.*

Remark. Actually the estimates hold for any Borel set E. Indeed by definition, $U_E = U_{\widetilde{E}}$ and

$$C_{\frac{2}{q},\, q'}\left(\frac{E\cap F_n(x,t)}{\sqrt{(n+1)t}}\right) \sim C_{\frac{2}{q},\, q'}\left(\frac{\widetilde{E}\cap F_n(x,t)}{\sqrt{(n+1)t}}\right).$$

Proof. The proof is same as in [23].
Let $\{E_k\}$ be a $\mathfrak{T}_q$-stratification of E. If $u \in \mathcal{V}_{mod}$ and $\mu = \operatorname{tr} u$ then $u_\mu = \sup u_{\mu_k}$ where $\mu_k = \mu\chi_{E_k}$. Hence $V_E = \sup V_{E_k}$. By proposition 5.25, $U_{E_k} = V_{E_k}$. These facts and Proposition 5.20(c) implies $U_E = V_E$. Since U_{E_k} satisfies the capacitary estimates (5.15) and

$$C_{\frac{2}{q},\, q'}\left(\frac{E_k\cap F_n(x,t)}{\sqrt{(n+1)t}}\right) \to C_{\frac{2}{q},\, q'}\left(\frac{\widetilde{E}\cap F_n(x,t)}{\sqrt{(n+1)t}}\right) \quad \text{as } n\to\infty.$$

it follows that U_E satisfies the corresponding capacitary estimates. $\square$

5.4 Localization

Definition 5.22 *Let A be a Borel subset of $\mathbb{R}^N$, we denote by $[u]_A$ the supremum of the $v \in \mathcal{U}_+(Q_T)$ which are dominated by u and vanishes on $\widetilde{A}^c$.*

We note here that $[u]_A = u \wedge U_A$

Lemma 5.23 *If $G \subset \mathbb{R}^N$ is a $\mathfrak{T}_q$-open set and $u \in \mathcal{U}_G(Q_T)$, then*

$$u = \sup\{v \in \mathcal{U}_G(Q_T) :\ v \leq u,\ v \text{ vanishes on an open neighborhood of } G\}.$$

Proof. Set $A = G^c$ and let $\{A_n\}$ be a sequence of closed subsets of A, such that $C_{\frac{2}{q},\, q'}(A\setminus A_n) \to 0$. By Proposition 5.20 we have

$$U_A \leq U_{A_n} + U_{A\setminus A_n},$$

thus

$$u = u\wedge U_A \leq u\wedge U_{A_n} + u\wedge (U_{A\setminus A_n}).$$

By Proposition 5.17, we have

$$U_{A\setminus A_n} \to 0.$$

Thus

$$u = \lim_{n\to\infty} u \wedge U_{A_n},$$

and the result follows. □

The next result points out the set-regularity of the correspondence $E \mapsto [u]_E$.

Proposition 5.24 *Let $u \in \mathcal{U}_+(Q_T)$.*
(i) If E is $\mathfrak{T}_q$-closed then,

$$\begin{aligned} [u]_E &= \inf\{[u]_D : \ E \subset D, \ D \text{ open}\}. &(5.32)\\ &= \sup\{[u]_F : \ F \subset E, \ F \text{ closed}\}. &(5.33)\end{aligned}$$

(ii) If E, F are two Borel sets then

$$[u]_E \le [u]_{F\cap E} + [u]_{E\setminus F}, \tag{5.34}$$

and

$$[[u]_E]_F = [[u]_F]_E = [u]_{F\cap E}. \tag{5.35}$$

(iii) Let E, F_n, $n = 1, 2, \ldots$ be Borel sets and let u be a positive solution of (3.1). If either $C_{\frac{2}{q},q'}(E \triangle F_n) \to 0$ or $\widetilde{F}_n \downarrow \widetilde{E}$ then

$$[u]_{F_n} \to [u]_E.$$

Proof. The proof uses a similar argument as in [23].
(i) Let $\mathcal{D} = \{D\}$ be the family of sets in (5.32). By (5.28) (with respect to the family $\mathcal{D}$)

$$\inf(u, U_E) = \inf(u, \inf_{D\in\mathcal{D}} U_D) = \inf_{D\in\mathcal{D}} \inf(u, U_D) \ge \inf_{D\in\mathcal{D}} [u]_D. \tag{5.36}$$

Obviously

$$[u]_{D_1} \wedge [u]_{D_2} \ge [u]_{D_1\cap D_2},$$

thus we can apply Proposition 3.4 and obtain that the function $v := \inf_{D\in\mathcal{D}}[u]_D$ is a solution of (3.1). Hence (5.36) implies $[u]_E \ge v$. The opposite inequality is obvious.

For the equality (5.33), Firstly, we note that the set $\{v \in \mathcal{U}_+(Q_T) : \ u \le u, \ \mathfrak{T}_q\text{-supp}\,(v) \subset^q E\}$ is closed under $\vee$. Thus, by Proposition 3.4, there exists an increasing sequence $\{v_n\}$ such that $v_n \approx_{E^c} = 0$ and $\lim_{n\to\infty} v_n = [u]_E$. Since v_n is an increasing sequence by Proposition 5.23 we can construct an increasing sequence $\{w_n\}$ such that each w_n vanishes on an open neighborhood B_n of E, $B_n \subset B_{n+1}$ and $\lim_{n\to\infty} w_n = [u]_E$. Now set $K_n = B_n^c$, then

$$w_n \le [u]_{K_n} \le [u]_E.$$

Letting n tend to infinity, we obtain the desired result.
(ii) Let $v \in \mathcal{U}_+(Q_T)$, $v \le u$ and $\mathfrak{T}_q\text{-supp}\,(v) \subset E$. Let D and D' be open sets such that

$\widetilde{E\cap F} \subset D$ and $\widetilde{E\setminus F} \subset D'$. By Lemma 2.8-[19], there exists a unique solution v_j^1, where $\frac{1}{[T]} < j \in \mathbb{N}$, of the problem

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u &= 0, && \text{in } \mathbb{R}^N \times (\frac{1}{j}, T] \\ u(., \frac{1}{j}) &= \chi_D(.)v(., \frac{1}{j}) && \text{in } \mathbb{R}^N. \end{aligned}$$

Also we consider v_j^2 and v_j^3 the unique solutions of the above problem with initial data $\chi_{D'}(x)v(x, \frac{1}{j})$ and $\chi_{(D_1\cup D_2)^c}$. In view of the proof of Proposition 5.20 we can prove that $v \le v_j^1 + v_j^2 + v_j^3$. By standard arguments there exists a subsequence, say $\{v_j^i\}$, $i = 1, 2, 3$, such that $v_j^i \to v^i$ and $v \le v^1 + v^2 + v^3$. Since v vanishes outside of E, it vanishes outside of $(D_1 \cup D_2)$, consequently $v(x, \frac{1}{j})\chi_{\chi_{(D_1\cup D_2)^c}} \to 0$, as $j \to \infty$, which implies $v_j^3 \to 0$. Thus we have

$$v \le v^1 + v^2 \le [u]_D + [u]_{D'}.$$

By (5.32) we have

$$v \le [u]_{F\cap E} + [u]_{E\setminus F},$$

since $v \in \{w \in \mathcal{U}_+(Q_T) : \ w \le u, \ \mathfrak{T}_q\text{-supp}\,(w) \subset^q E\}$ is arbitrary the result follows in the case where E is closed. In the general case the result follows by (5.33).

Put $A = \widetilde{E}$ and $B = \widetilde{F}$. It follows directly from the definition that

$$[[u]_A]_B \le \inf(u, U_A, U_B).$$

The largest solution dominated by u and vanishing on $A^c \cup B^c$ is $[u]_{A\cap B}$. Thus

$$[[u]_A]_B \le [u]_{A\cap B}.$$

On the other hand

$$[u]_{A\cap B} = [[u]_{A\cap B}]_B \le [[u]_A]_B,$$

this proves (5.35).

(iii) By (5.34)

$$[u]_E \le [u]_{F_n\cap E} + [u]_{E\setminus F_n}, \qquad [u]_{F_n} \le [u]_{F_n\cap E} + [u]_{F_n\setminus E}.$$

If $C_{\frac{2}{q}, q'}(E\triangle F_n) \to 0$, then by Proposition (5.17)(c) we have that $U_{E\triangle F_n} \to 0$. Since $[u]_{E\setminus F_n}$, $[u]_{F_n\setminus E} \le U_{E\triangle F_n}$, the result follows by the above inequalities, if we let n go to infinite.

If $\widetilde{F}_n \downarrow \widetilde{E}$. By Proposition (5.17)(c) we have $U_{E_n} \to U_E$, thus

$$[u]_E \le \lim_{n\to\infty}[u]_{F_n} = \lim_{n\to\infty} u \wedge U_{F_n} \le \lim_{n\to\infty} \inf(u, U_{F_n}) \le \inf(u, U_E).$$

And since $[u]_E$ is the largest solution under $\inf(u, U_E)$ and the function $v = \lim_{n\to\infty}[u]_{F_n}$ is a solution of (3.1), we have that $U_E \le v$, and the proof of (5.34) is complete. $\square$

Definition 5.25 *Let μ be a positive Radon measure on $\mathbb{R}^N$ which vanishes on compact sets of $C_{\frac{2}{q},q'}$-capacity zero.*
(a) The $\mathfrak{T}_q$-support of μ, denoted $\mathfrak{T}_q$-supp(μ), is the intersection of all $\mathfrak{T}_q$-closed sets F such that $\mu(F^c) = 0$.
(b) We say that μ is concentrated on a Borel set E if $\mu(E^c) = 0$.

Proposition 5.26 *If μ is a measure as in the previous definition then,*

$$\mathfrak{T}_q\text{-}supp\,(\mu) \sim^q \mathfrak{T}_q\text{-}supp\,(u_\mu).$$

Proof. Put $F = \text{supp}^q u_\mu$. By Proposition 5.19(iv) u_μ vanishes on F^c and by Proposition 5.23(c) there exists an increasing sequence of positive solutions u_n such than each function u_n vanishes outside a closed subset F, say F_n, and $u_n \uparrow u_\mu$. If $S_n := \mathfrak{T}_q$-supp$\,(u_n)$ then $S_n \subset F_n$ and $\{S_n\}$ increases. Thus $\{\overline{S}_n\}$ is an increasing sequence of closed subsets of F and, setting $\mu_n = \mu\chi_{\overline{S}_n}$, we find $u_n \leq u_{\mu_n} \leq u_\mu$ so that $u_{\mu_n} \uparrow u_\mu$. This, in turn, implies

$$\mu_n \uparrow \mu, \qquad \mathfrak{T}_q\text{-supp}(\mu) \subset^q \widetilde{\bigcup_{n=1}^{\infty} \overline{S}_n} \subset F.$$

If D is an open set and $\mu(D) = 0$ it is clear that u_μ vanishes on D. Therefore u_{μ_n} vanishes outside of $\overline{S}_n$, thus outside $\mathfrak{T}_q$-supp$\,(\mu)$. Consequently u_μ vanishes outside $\mathfrak{T}_q$-supp(μ), i.e., $F \subset^q \mathfrak{T}_q$-supp$\,(\mu)$.

Second proof. The result follows by Proposition 5.7 and Definition 5.8 $\square$

Definition 5.27 *Let u be a positive solution and A a Borel set. Put*

$$[u]^A := \sup\{[u]_F : \; F \subset^q A, \; F \; q\text{−closed}\}.$$

Definition 5.28 *Let $\beta > 0$, $u \in C(Q_T)$. For any Borel set A we denote by u_β^A the positive solution of*

$$\begin{aligned} \partial_t v - \Delta v + |v|^{q-1}v &= 0 && \text{in } \mathbb{R}^N \times (\beta, \infty) \\ v(.,\beta) &= \chi_A(.)u(.,\beta) && \text{in } \mathbb{R}^N. \end{aligned}$$

Proposition 5.29 *Let u be a positive solution of (3.1) and put $E = \mathfrak{T}_q$-supp$\,(u)$.*
(i) If D is a $\mathfrak{T}_q$-open set such that $E \subset^q D$, then

$$[u]^D = \lim_{\beta \to 0} u_\beta^D = [u]_D = u. \tag{5.37}$$

(ii) If A is a $\mathfrak{T}_q$-open set, then

$$u \approx_A 0 \Leftrightarrow u^Q = \lim_{\beta \to 0} u_\beta^Q = 0, \qquad \forall Q \; \mathfrak{T}_q\text{−open} : \; \widetilde{Q} \subset^q A. \tag{5.38}$$

(iii) Finally,

$$u \approx_A 0 \Leftrightarrow [u]^A = 0. \tag{5.39}$$

Proof. The proof is similar as in the one as in [23]
Case 1: E is closed. Since u vanishes in E^c, it yields $u \in C(Q_\infty \cup E^c)$ and $u = 0$ on E^c. If, in addition, D is an open neighborhood of E, we have

$$\lim_{t\to 0}\int_{E^c}\phi(x)u(x,t)dx = 0, \qquad \forall \phi \in C_0(E^c).$$

Thus,

$$\lim_{\beta\to 0} u_\beta^{D^c} = 0.$$

Since

$$u_\beta^D \le u \le u_\beta^D + u_\beta^{D^c}, \qquad \forall t \ge \beta,$$

it follows

$$u = \lim_{\beta\to 0} u_\beta^D. \tag{5.40}$$

If we assume that D is $\mathfrak{T}_q$-open and $E \subset^q D$ then, for every $\varepsilon > 0$, there exists an open set O_ε such that $D \subset O_\varepsilon,\ E \subset O_\varepsilon$ and $C_{\frac{2}{q},\,q'}(O'_\varepsilon) < \varepsilon$ where $O'_\varepsilon = O_\varepsilon \setminus D$. Therefore

$$u_\beta^{O_\varepsilon}(x,t) - u_\beta^D(x,t) \le U_{O'_\varepsilon}(x, t-\beta), \qquad \forall t \ge \beta.$$

We note here that $\lim_{\varepsilon\to 0} U_{O'_\varepsilon}(x,t-\beta) = 0$ holds uniformly with respect to β. Since $\lim_{\beta\to 0} u_\beta^{O_\varepsilon}(x,t) = u$ it follows that $u = \lim_{\beta\to 0} u_\beta^D$. The same arguments shows that $\lim_{\beta\to 0} u_\beta^{D^c} = 0$. Thus we have

$$u = \lim_{\beta\to 0} u_\beta^D \le [u]_D \le u.$$

Hence $u = [u]_D$. By Lemma 2.7, there exists a $\mathfrak{T}_q$-open set Q such that $E \subset^q Q \subset \widetilde{Q} \subset^q D$, then $u = [u]_Q \le [u]^D$. Hence $u = [u]^D$.

In addition, there holds $E \subset^q A^c \subset^q \widetilde{Q}^c$. Thus the direction "$\Rightarrow$" in (5.38) follows by the previous argument if we replace D by $\widetilde{Q}^c$. For the opposite direction, by Proposition 2.38, for any $\xi \in A$, there exists a $\mathfrak{T}_q$-open set O_ξ such that $\widetilde{O}_\xi \subset^q A$. Using (i) we infer $u = \lim_{\beta\to 0} u_\beta^{\widetilde{O}_\xi^c}$. Finally, since $u_\beta^{\widetilde{O}_\xi^c} \approx_{Q_\xi} 0$ for all $\beta > 0$, it implies $u \approx_{O_\xi} 0$ by Proposition 5.17(i), and the result follows in this case by Proposition 5.19(iv).
Case 2. Assume E is $\mathfrak{T}_q$-closed. Let $\{E_n\}$ be a $\mathfrak{T}_q$-stratification of E such that $C_{\frac{2}{q},\,q'}(E\backslash E_n) \to 0$. If D is a $\mathfrak{T}_q$-open such that $E \subset^q D$ then, by the first case we have,

$$\lim_{\beta\to 0}([u]_{E_n})_\beta^D = [u]_{E_n}. \tag{5.41}$$

By (5.34) and the definition of u_β^D, and since $[u]_E = u$,

$$u_\beta^D = ([u]_E)_\beta^D \le ([u]_{E\cap E_n})_\beta^D + \left([u]_{E\backslash E_n}\right)_\beta^D = ([u]_{E_n})_\beta^D + \left([u]_{E\backslash E_n}\right)_\beta^D. \tag{5.42}$$

Let $\{\beta_k\}$ be a decreasing sequence converging to 0 such that the following limits exist

$$w := \lim_{k\to\infty} u_{\beta_k}^D, \qquad w_n = \lim_{k\to\infty}\left([u]_{E\backslash E_n}\right)_{\beta_k}^D, \qquad n = 1, 2, \dots .$$

Then by (5.41) and (5.42),

$$[u]_{E_n} \leq w \leq [u]_{E_n} + w_n \leq [u]_{E_n} + U_{E\setminus E_n}.$$

Further, by (5.33) and Proposition 5.20(c)

$$[u]_{E_n} \to [u]_E = u, \qquad U_{E\setminus E_n} \to 0.$$

Hence $w = u$. This implies (5.40), which in turn implies (5.37).
To verify (5.38) in the direction $\Rightarrow$ we apply (5.42) with D replaced by Q. We obtain

$$([u]_E)^Q_\beta \leq ([u]_{E_n})^Q_\beta + \left([u]_{E\setminus E_n}\right)^Q_\beta.$$

By the first case we have

$$\lim_{\beta\to 0} ([u]_{E_n})^Q_\beta = 0.$$

There exists a decreasing sequence converging to 0, still denoted by $\{\beta_k\}$, such that the following limits exist

$$\lim_{k\to\infty} u^Q_{\beta_k}, \qquad \lim_{k\to\infty} \left([u]_{E\setminus E_n}\right)^Q_{\beta_k}, \qquad n = 1, 2, \dots .$$

Then

$$\lim_{k\to\infty} u^Q_{\beta_k} \leq \lim_{k\to\infty} \left([u]_{E\setminus E_n}\right)^Q_{\beta_k} \leq U_{E\setminus E_n},$$

since $U_{E\setminus E_n} \to 0$ we obtain (5.38) in the direction $\Rightarrow$. The assertion in the opposite direction is proved as in Case 1. This complete the proofs of (i) and (ii).

Finally we prove (iii). First assume that $u \approx_A 0$. If F is a $\mathfrak{T}_q$-closed set such that $F \subset^q A$, then by Lemma 2.7 there exists a $\mathfrak{T}_q$-open set $\mathfrak{T}_q$ such that $F \subset^q Q \subset \widetilde{Q} \subset^q A$. Therefore, applying (5.37) to $v := [u]_F$ and using (5.38) we obtain

$$v = \lim_{\beta\to 0} v^Q_\beta \leq \lim_{\beta\to 0} u^Q_\beta = 0.$$

By definition of $[u]^A$, this implies $[u]^A = 0$.

If $[u]^A = 0$, then for any $\mathfrak{T}_q$-open set $Q \subset \widetilde{Q} \subset^q A$ there holds $[u]_Q = 0$.. Now since $\mathfrak{T}_q$-supp $(u^Q_\beta) \subset^q \widetilde{Q}$ we have for some subsequence $\beta_k \downarrow 0$, $\lim_{k\to\infty} u^Q_{\beta_k} \leq [u]_Q = 0$. Thus $u \approx_Q 0$ by (5.38). Applying once again Proposition 2.38 and Proposition 5.19(iv) we conclude $u \approx_A 0$. □

Definition 5.30 *Let $u,\ v \in \mathcal{U}_+(Q_T)$ and let A be a $\mathfrak{T}_q$-open set. We say that $u = v$ on A if $u \ominus v$ and $v \ominus u$ vanishes on A. This relation is denoted by $u \approx_A v$.*

Proposition 5.31 *Let $u,\ v \in \mathcal{U}_+(Q_T)$ and let A be a $\mathfrak{T}_q$-open set. Then,*
(i)

$$u \approx_A v \Leftrightarrow \lim_{\beta\to 0} |u - v|^Q_\beta = 0. \tag{5.43}$$

for every $\mathfrak{T}_q$-open set Q such that $\widetilde{Q} \subset^q A$.
(ii)

$$u \approx_A v \Leftrightarrow [u]_F = [v]_F, \tag{5.44}$$

for every $\mathfrak{T}_q$-closed set F such that $F \subset^q A$.

Proof. The proof is similar, but in a parabolic framework, to the elliptic one in [23].
By definition $u \approx_A v$ is equivalent to $u \ominus v \approx_A = 0$ and $v \ominus u \approx_A = 0$. Hence, by (5.38) we have $w_\beta = (u \ominus v)_\beta^Q \to_{\beta \to 0} 0$. Set $f_\beta = ((u - v)_+)_\beta^Q$ and consider the problem

$$\begin{aligned} \partial_t w - \Delta w + |w|^q &= 0, \qquad \text{in } B_j(0) \times (\beta, \infty) \\ w &= 0, \qquad \text{on } \partial B_j(0) \times (\beta, \infty) \\ w(., \beta) &= \mu, \qquad \text{in } B_j(0). \end{aligned}$$

Let w_j and f_j be solutions of the above problem, with initial data $\chi_Q(u \ominus v)(x, \beta)$ and $\chi_Q(u - v)_+(x, \beta)$. By [19, Lemma 2.7], the sequences $\{w_j\}$ and $\{f_j\}$ are increasing. Also, we recall that $u \ominus v$ is the smallest solution which dominates the subsolution $(u - v)_+$, thus $w_j \geq v_j$, $\forall j \in \mathbb{N}$. Furthermore, in view of [19, Lemma 2.8], there holds $\lim_{j \to \infty} w_j = w_\beta$ and $\lim_{j \to \infty} f_j = f_\beta$. Thus $w_\beta \geq f_\beta$, and letting $\beta \to 0$ we derive

$$((u - v)_+)_\beta^Q \to 0.$$

By the same argument we have

$$((v - u)_+)_\beta^Q \to 0,$$

this implies (5.43) in the direction $\Rightarrow$.

For the opposite direction, we consider the problem

$$\begin{aligned} \partial_t w - \Delta w + |w|^q &= 0, \qquad \text{in } B_j(0) \times (\beta, \infty) \\ w &= h, \qquad \text{on } \partial B_j(0) \times (\beta, \infty) \\ w(., \beta) &= \mu, \qquad \text{in } B_j(0). \end{aligned}$$

Let $Q \subset \widetilde{Q} \subset^q A$ be a $\mathfrak{T}_q$-open set and w_j be the solution of the above problem, with $h = \chi_Q(|u - v|)$ and $\mu = \chi_Q|u - v|dx$. Also, let f_j be the solution of the above problem with $h = \chi_{Q^c}|u - v|$ and $\mu = \chi_{Q^c}|u - v|dx$, then

$$|u - v| \leq w_j + f_j.$$

In view of [19, Lemma 2.8], there exist subsequences, say $\{w_j\}$ and $\{f_j\}$, satisfying $\lim_{j \to \infty} w_j = w$ and $\lim_{j \to \infty} f_j = f$, such that (w, f) solves the problem

$$\begin{aligned} \partial_t v - \Delta v + |v|^{q-1} v &= 0, \qquad \text{in } \mathbb{R}^N \times (\beta, \infty) \\ v(., \beta) &= \mu \qquad \text{in } \mathbb{R}^N, \end{aligned}$$

with initial data $\mu = \chi_Q|u - v|dx$ and $\mu = \chi_{Q^c}|u - v|dx$ respectively. By uniqueness of the problem (see [19, Lemma 2.8]), we have $w = |u - v|_\beta^Q$ and $f = |u - v|_\beta^{Q^c}$. Let β_k be a decreasing sequence such that the following limit exists

$$\lim_{k \to \infty} |u - v|_{\beta_k}^{Q^c}.$$

Since $\lim_{\beta \to 0} |u - v|_\beta^Q = 0$, we have

$$|u - v| \leq \lim_{k \to \infty} |u - v|_{\beta_k}^{Q^c}.$$

Now since $|u-v|_{\beta_k}^{Q^c} \approx_Q 0$, by Proposition 5.17(i) we have $\lim_{k\to\infty}|u-v|_{\beta_k}^{Q^c} \approx_Q = 0$. Using the fact that $u \ominus v$ is the smallest solution which dominates the subsolution $(u-v)_+$, there holds $\max\{u\ominus v,\ v\ominus u\} \leq \lim_{k\to\infty}|u-v|_{\beta_k}^{Q^c}$ and the result follows in this case by Propositions 5.23 and 5.19(iv).
(ii) We assume that $u \approx_A v$.
For any two positive solutions u, v we have

$$u + (v-u)_+ \leq v + (u-v)_+ \leq v + u \ominus v \tag{5.45}$$

If F is a $\mathfrak{T}_q$-closed set and Q a $\mathfrak{T}_q$-open set such that $F \subset^q Q$, we claim that

$$[u]_F \leq [v]_Q + [u\ominus v]_Q. \tag{5.46}$$

To verify this inequality, we observe first that (see (5.34))

$$u = [u]_{\mathbb{R}^N} \leq [u]_Q + [u]_{Q^c},$$

thus by (5.45)

$$[u]_F \leq [u]_{\mathbb{R}^N} \leq v + u\ominus v \leq [v]_Q + [v]_{Q^c} + [u\ominus v]_Q + [u\ominus v]_{Q^c}.$$

The subsolution $w := \left([u]_F - ([v]_Q + [u\ominus v]_Q)\right)_+$ is dominated by the supersolution $[u\ominus v]_{Q^c} + [v]_{Q^c}$. By definition we have

$$w \leq [w]_\dagger \leq [u\ominus v]_{Q^c} \oplus [v]_{Q^c} \leq [u\ominus v]_{Q^c} + [v]_{Q^c}.$$

Thus $[w]_\dagger \approx_Q 0$. But $w \leq [u]_F$ which implies $[w]_\dagger \leq [u]_F$, that is $\mathfrak{T}_q\text{-supp}\,([w]_\dagger) \subset^q F \subset^q Q$. Taking into account that $[w]_\dagger \approx_Q 0$ we have that $w = [w]_\dagger = 0$ and the proof of (5.46) is completed.
If we choose a $\mathfrak{T}_q$-open set Q such that $F \subset^q Q \subset \widetilde{Q} \subset^q A$ (see Lemma 2.7), and using the fact that $u\ominus v \approx_A = 0 \Rightarrow [u\ominus v]_F = 0$ (see (5.39)) and (5.46), we infer

$$[u]_F \leq [v]_Q.$$

Now by Lemma 2.8(I), we can construct a decreasing sequence $\{Q_j\}$ of open sets such that $\cap Q_j \sim^q F$, thus by Proposition 5.24(iii) we have

$$[u]_F \leq \lim_{n\to\infty}[v]_{Q_n} = [v]_F.$$

Similarly, $[v]_F \leq [u]_F$ and hence the equality holds.

Next we assume that $[u]_F = [v]_F$ for any $\mathfrak{T}_q$-closed set $F \subset^q A$. If Q is a $\mathfrak{T}_q$-open set such that $F \subset^q Q \subset \widetilde{Q} \subset^q A$ (see Lemma 2.7), we have

$$u\ominus v \leq ([u]_Q \oplus [u]_{Q^c}) \ominus [v]_Q,$$

where in the last inequality we have used the fact that

$$u = [u]_{\mathbb{R}^N} \leq [u]_Q + [u]_{Q^c} \Rightarrow\ u \leq [u]_Q \oplus [u]_{Q^c} \leq [u]_Q + [u]_{Q^c}.$$

Since $([u]_Q \oplus [u]_{Q^c}) \ominus ([v]_Q)$ is the smallest solution dominating $(([u]_Q \oplus [u]_{Q^c}) - [v]_Q)_+$, we have

$$(([u]_Q \oplus [u]_{Q^c}) - [v]_Q)_+ \leq (([u]_Q + [u]_{Q^c}) - [v]_Q)_+ = [u]_Q + [u]_{Q^c} - [v]_Q = [u]_{Q^c},$$

since by assumption we have $[u]_Q = [v]_Q$. Thus we have

$$[u \ominus v]_F \leq u \ominus v \leq [u]_{Q^c},$$

This means $\mathfrak{T}_q$-supp$([u \ominus v]_F) \subset^q F$ and $[u \ominus v]_F \approx_Q 0$, which in turn implies $[u \ominus v]_F = 0$, and by 5.39 $u \ominus v \approx_A = 0$. Similarly, $v \ominus u \approx_A 0$. □

Corollary 5.32 *If A is a $\mathfrak{T}_q$-open set, the relation $\approx_A$ is an equivalence relation in $\mathcal{U}_+(Q_T)$.*

Proof. This is an immediate consequence of (5.43). □

6 The precise initial trace

6.1 The regular initial set

Lemma 6.1 *Let $u \in \mathcal{U}_+(Q_T)$ and Q be a $\mathfrak{T}_q$-open set. Then for any $\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support in $\widetilde{Q}^c$, we have*

$$\int_0^T \int_{\mathbb{R}^N} (u \wedge U_Q)^q(t,x)\mathbb{H}^{2q'}[\eta]_+ dxdt < \infty.$$

Proof. By Proposition 5.9 and the properties of U_Q, there holds

$$\lim_{t\to 0} \int_Q u \wedge U_Q(x,t)\eta_+^{2q'}(x)dx = 0,$$

and the result follows by the estimates in Lemma 5.11. □

Proposition 6.2 *Let $u \in \mathcal{U}_+(Q_T)$ and Q be a $\mathfrak{T}_q$-open set. We assume that $u \wedge U_Q$ is a moderate solution with initial data μ. Then for any $\xi \in Q$ there exists a $\mathfrak{T}_q$-open set $O_\xi \subset Q$ such that*

$$\int_0^T \int_{\mathbb{R}^N} u^q(t,x)\mathbb{H}^{2q'}[\chi_{O_\xi}]_+ dxdt < \infty.$$

Furthermore, for any $\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support in Q, we have

$$\lim_{t\to 0} \int_Q u(x,t)\eta_+^{2q'}(x)dx = \int_Q \eta^{2q'} d\mu.$$

Proof. Let $\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support in Q. Since $\eta_+^{2q'}$ is a quasi continuous function we have by Lemma 5.7 that

$$\lim_{t\to 0}\int_Q u \wedge U_Q(x,t)\eta_+^{2q'}(x)dx = \int_Q \eta^{2q'} d\mu.$$

Using the properties of U_{Q^c},

$$\lim_{t\to 0}\int_Q u \wedge U_{Q^c}(x,t)\eta_+^{2q'}(x)dx = 0.$$

Combining all above and using the fact that $u \leq u \wedge U_Q + u \wedge U_{Q^c}$ we get

$$\begin{aligned}
\int_Q \eta^{2q'} d\mu = \lim_{t\to 0}\int_Q u \wedge U_Q(x,t)\eta_+^{2q'}(x)dx \quad &\leq \quad \lim_{t\to 0}\int_Q u(x,t)\eta_+^{2q'}(x)dx \\
\leq \lim_{t\to 0}\int_Q u \wedge U_Q(x,t)\eta_+^{2q'}(x)dx \quad &+ \quad \lim_{t\to 0}\int_Q u \wedge U_{Q^c}(x,t)\eta_+^{2q'}(x)dx \\
&= \quad \int_Q \eta^{2q'} d\mu + 0.
\end{aligned}$$

In view of the proof of Lemma 5.2 and by 5.3 there holds

$$\int_0^T \int_{\mathbb{R}^N} (u \wedge U_Q)^q(t,x)\mathbb{H}^{2q'}[\eta]_+ dxdt < \infty, \tag{6.1}$$

for any $\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support in Q. By Lemma 4.2, there exists $\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $O_\xi \subset Q$ and $\mathfrak{T}_q\text{-supp}\,(\eta) \subset Q$. Thus we have by (6.1) and the properties of η,

$$\int_0^T \int_{\mathbb{R}^N} (u \wedge U_{Q^c})^q(t,x)\mathbb{H}^{2q'}[\chi_{O_\xi}]dxdt < \infty. \tag{6.2}$$

□

Definition 6.3 *(Section 10.1-[2]) Let Q be a Borel set. We denote $W^{\frac{2}{q},q'}(E^c)$ the closure of the space of C^∞ functions (with respect the norm $||\cdot||_{W^{\frac{2}{q},q'}}$) with compact support in E^c.*

Proposition 6.4 *Let u be a positive solution of (3.1) and Q a bounded $\mathfrak{T}_q$-open sets such that*

$$\int_0^T \int_{\mathbb{R}^N} u^q(t,x)\mathbb{H}^{2q'}[\chi_Q]dxdt < \infty. \tag{6.3}$$

(i) Then, there exists an increasing sequence of $\mathfrak{T}_q$-open set $\{Q_n\}$ satisfying $Q_n \subset Q$, $\widetilde{Q}_n \subset^q Q_{n+1}$ and $Q_0 := \bigcup_{n=1}^\infty Q_n \sim^q Q$, such that the solution $v_n = u \wedge Q_n$ is moderate, $v_n \uparrow [u]_Q$, $\mathrm{tr}(v_n) \to \mu_Q$.

(ii) For any $\eta \in W^{\frac{2}{q},q'}(Q)$ we have

$$\lim_{t\to 0}\int_Q u(x,t)\eta_+^{2q'}(x)dx = \int_Q \eta_+^{2q'}(x)d\mu_Q.$$

Proof. We choose a point $z \in Q$. Then by Lemma 4.2 there exist a $\mathfrak{T}_q$-open set V, such that $z \in V \subset \widetilde{V} \subset Q$, and a function $\psi \in W^{\frac{2}{q},q'}(\mathbb{R}^N)$ such that $\psi = 1$ q.a.e. on V and $\psi = 0$ outside Q. By Lemma 2.38, there exists a $\mathfrak{T}_q$-open set $z \in O_z \subset \widetilde{O}_z \subset V$.

We assert that the function

$$v_z = u \wedge U_{O_z} \tag{6.4}$$

is a moderate solution.

Indeed, let $B_R(0)$ be a ball with radius R large enough such that $Q \subset\subset B_R(0)$. Also, let $0 \leq \eta \leq 1$ be a smooth function with compact support in $B_{2R}(0)$ and $\eta = 1$ on $B_R(0)$. Then the function $\zeta = (1-\psi)\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with compact support in $B_{2R}(0) \setminus \widetilde{V}$. Now

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^N} v_z^q(t,x)\mathbb{H}^{2q'}[\chi_{B_R(0)}]dxdt &\leq \int_0^T \int_{\mathbb{R}^N} v_z^q(t,x)\mathbb{H}^{2q'}[\psi]dxdt + \int_0^T \int_{\mathbb{R}^N} v_z^q(t,x)\mathbb{H}^{2q'}[1-\psi]dxdt \\ &\leq \int_0^T \int_{\mathbb{R}^N} v_z^q(t,x)\mathbb{H}^{2q'}[\psi]dxdt + \int_0^T \int_{\mathbb{R}^N} v_z^q(t,x)\mathbb{H}^{2q'}[\zeta]dxdt < \infty,\end{aligned}$$

where the first integral in the last inequality is finite by assumption and the second integral is finite by Lemma 6.1. Thus since $B_R(0)$ is arbitrary, the function $u \wedge O_z$ is a moderate solution.

By the *quasi-Lindelöff property* there exists a non decreasing sequence of $\mathfrak{T}_q$- open set $\{O_n\}$ such that $Q \sim^q \cup O_n$ and (by the above arguments) the solution $u \wedge U_{O_n}$ is moderate for any $n \in \mathbb{N}$. Now, by Lemma 2.8 (II)(i)-(ii), for any $n \in \mathbb{N}$, there exists an increasing sequence $\{A_{n,j}\}$ of $\mathfrak{T}_q$-open sets such that $\widetilde{A}_{n,j} \subset^q A_{n,j+1} \subset^q E_n$ and $\bigcup_{j=1}^\infty A_{n,j} \sim^q E_n$. Put

$$Q_n = \bigcup_{k+j=n} A_{k,j}.$$

Then

$$\widetilde{Q}_n \subset \bigcup_{k+j=n} \widetilde{A}_{k,j} \subset^q \bigcup_{k+j=n} \widetilde{A}_{k,j+1} = Q_{n+1}.$$

Hence,

$$Q_0 := \bigcup Q_n \sim^q Q.$$

Now, we will prove that $v_n = u \wedge U_{Q_n} \to u \wedge U_Q$. By Proposition 5.24(ii) we have

$$u \wedge U_Q \leq u \wedge U_{Q_n} + u \wedge U_{Q \setminus Q_n}.$$

Since $Q \setminus Q_n \downarrow F$ with $C_{\frac{2}{q},\, q'}(F) = 0$, we have by Proposition 5.24(iii) that

$$u \wedge U_{Q \setminus Q_n} \to 0.$$

The opposite inequality is obvious and the result follows in this assertion. By Lemma 5.24(ii) $v_n = [v_{n+k}]_{Q_n}$, $\forall k \in \mathbb{N}$. Therefore

$$\mu_n(Q_n) = \mu_{n+k}(Q_n) = \mu_Q(Q_n). \tag{6.5}$$

(ii) First we assume that the function $\eta \in W^{\frac{2}{q},q'}(Q)$ has compact support in Q. Then by Lemma 4.3 there exists a function η_k such that $\mathfrak{T}_q$-supp $(\eta_k) \subset Q_k$, and

$$||\eta - \eta_k||_{W^{\frac{2}{q},q'}} \leq \frac{1}{k}, \tag{6.6}$$

and $|\eta_k| \leq |\eta|$. By Lebesgue's dominated theorem, we can assume that η_k satisfies

$$\int_0^T \int_{\mathbb{R}^N} u^q(t,x)(\mathbb{H}[\eta - \eta_k])^{2q'} dxdt < \frac{1}{k}$$

Also in view of Proposition 5.2 and (5.7)-(5.11),

$$\lim_{t\to 0} \int_Q u(x,t)\eta^{2q'}(x)dx \leq C||\eta||^{q'}_{L^\infty(\mathbb{R}^N)}||\eta||^{q'}_{W^{\frac{2}{q},q'}} + \int_0^T \int_{\mathbb{R}^N} u^q(t,x)(\mathbb{H}[\eta])^{2q'} dxdt,$$

But by (6.5) and Lemma 6.2 we have

$$\begin{aligned}
\left(\int_Q \eta_k^{2q'}(x)d\mu_Q\right)^{\frac{1}{2q'}} &= \lim_{t\to 0}\left(\int_Q u(x,t)\eta_k^{2q'}(x)dx\right)^{\frac{1}{2q'}} \\
&\leq \lim_{t\to 0}\left(\int_Q u(x,t)\eta^{2q'}(x)dx\right)^{\frac{1}{2q'}} \\
&\leq \lim_{t\to 0}\left(\int_Q u(x,t)\left(\eta - \eta_k\right)^{2q'}(x)dx\right)^{\frac{1}{2q'}} + \lim_{t\to 0}\left(\int_Q u(x,t)\eta_k^{2q'}(x)dx\right)^{\frac{1}{2q'}} \\
&\leq \left(\int_Q \eta_k^{2q'}(x)d\mu_Q\right)^{\frac{1}{2q'}} + C||\eta - \eta_k||^{\frac{1}{2}}_{L^\infty(\mathbb{R}^N)}||\eta - \eta_k||^{\frac{1}{2}}_{W^{\frac{2}{q},q'}} \\
&+ \left(\int_0^T \int_{\mathbb{R}^N} u^q(t,x)(\mathbb{H}[\eta - \eta_k])^{2q'} dxdt\right)^{\frac{1}{2q'}} \\
&\leq \left(\int_Q \eta_k^{2q'}(x)d\mu_Q\right)^{\frac{1}{2q'}} + C\frac{1}{\sqrt{k}}||\eta||^{\frac{1}{2}}_{L^\infty(\mathbb{R}^N)} + \left(\frac{1}{k}\right)^{\frac{1}{2q'}}.
\end{aligned}$$

The result follows in this case by letting $k \to \infty$.
For the general case, by theorem 10.1.1 in [2], there exists a function η_k with compact support in Q such that

$$||\eta - \eta_k||_{W^{\frac{2}{q},q'}} \leq \frac{1}{k}, \tag{6.7}$$

and $|\eta_k| \leq |\eta|$. The result follows as above. □

Remark. By Lemma 6.2 and (6.4), we have that the definition of the regular points in the elliptic case (see [23]) coincides with our definition of the regular points.

Lemma 6.5 *Let Q be a $\mathfrak{T}_q$-open set and $u \in \mathcal{U}_+(Q_T)$ satisfy (6.3). Then*
i)

$$[u]_Q = \sup\{[u]_F :\ F \subset^q Q,\ F\ \mathfrak{T}_q\text{−closed}\}. \tag{6.8}$$

ii) For every $\mathfrak{T}_q$-open set $O \subset \widetilde{O} \subset^q Q$ such that $[u]_O$ is a moderate solution we have

$$\mu_Q \chi_{\widetilde{O}} = \mathrm{tr}'[u]_O) = \mathrm{tr}([u_Q]_O). \tag{6.9}$$

Finally, μ_Q is $\mathfrak{T}_q$-locally finite on Q and σ-finite on $Q' := \cup Q_n$.
iii) If $\{w_n\} \subset \mathcal{U}_+(Q_T)$ is a nondecreasing sequence of moderate solutions such that $\mathfrak{T}_q$-supp$(w_n) \subset^q Q$ and $w_n \uparrow [u]_Q$, then $\mathrm{tr}(w_n) = \nu_n \uparrow \mu_Q$.

Proof. i) Let u^* denote the right-hand side of (6.8). By Proposition 3.4 there exists a nondecreasing sequence $\{[u]_{F_n}\}$ such that $[u]_{F_n} \uparrow u^*$. We consider the function $[u]_{Q_n}$ of Proposition 6.4. Then by Proposition 5.24 we have

$$[u]_{F_n} \leq [u]_{F_n \cap Q_m} + [u]_{F_n \setminus Q_m}.$$

Now we note that $F_n \setminus Q_m$ is a $\mathfrak{T}_q$-closed set and $\cap_{m=1}^{\infty} F_n \setminus Q_m = A$ with $C_{\frac{2}{q},\, q'}(A) = 0$. Thus by Proposition 5.19 we have that $\lim_{m\to\infty} U_{F_n \setminus Q_m} = 0$ which implies $\lim_{m\to\infty}[u]_{F_n \setminus Q_m} = 0$. Thus $[u]_{F_n} \leq \lim [u]_{Q_m} = u_Q$. Letting $n \to \infty$ we have $u^* \leq u_Q$. By definition of u^* we have that $u_Q \leq u^*$, thus $u^* = u_Q$.

ii) Put $\mu_O = \mathrm{tr}([u]_O)$. If F is a $\mathfrak{T}_q$-closed set such that $F \subset^q O$, by Proposition 5.24-(ii) we have

$$\mathrm{tr}([u]_F) = \mathrm{tr}([[u]_O]_F) = \mu_O \chi_F. \tag{6.10}$$

In particular the compatibility condition holds: if $O' \subset \widetilde{O}' \subset^q Q$ is $\mathfrak{T}_q$-open set such that $[u]_{O'}$ is a moderate solution

$$\mu_{O \cap O'} = \mu_O \chi_{\widetilde{O} \cap \widetilde{O}'} = \mu_{O'} \chi_{\widetilde{O} \cap \widetilde{O}'}. \tag{6.11}$$

With the notation of (6.5), $[v_{n+k}]_{Q_k} = v_k$ and hence $\mu_{n+k} \chi_{\widetilde{Q}_k} = \mu_k$ for every $k \in \mathbb{N}$.

Since $[u]_F$ is moderate, we have by (6.11)

$$[v_n]_F = [u]_{F \cap \widetilde{Q}_n} \uparrow [u]_F. \tag{6.12}$$

In addition, $[u_Q]_F \geq \lim_{n\to\infty} [v_n]_F = [u]_F$, jointly with $u_Q \leq u$, leads to,

$$[u]_F = [u_Q]_F. \tag{6.13}$$

By (6.10) and (6.12), if F is a $\mathfrak{T}_q$-closed subset of $\mathcal{R}_q(u)$ and $[u]_F$ is moderate,

$$\mathrm{tr}([u]_F) = \lim_{n\to\infty} \mathrm{tr}([v_n]_F) = \lim_{n\to\infty} \mu_n \chi_F = \mu_{\mathcal{R}_q} \chi_F, \tag{6.14}$$

which implies (6.9).

Since $Q' := \cup Q_n \sim^q Q$, μ_Q is σ-finite on Q'. The assertion that μ_Q is $\mathfrak{T}_q$-locally finite on Q is a consequence of the fact that every point in Q is contained in a $\mathfrak{T}_q$-open set $O \subset^q \widetilde{O} \subset Q$ such that $[u]_O$ is a moderate solution (see (6.4)).

iii) If w is a moderate solution and $w \leq u_Q$ and $\mathfrak{T}_q$-supp $(w) \subset^q Q$, then $\tau := \mathrm{tr}(w) \leq \mu_Q$. Indeed

$$[w]_{Q_n} \leq [u]_{Q_n} = v_n,\ [w]_{Q_n} \uparrow w \Rightarrow \mathrm{tr}([w]_{Q_n}) \uparrow \tau \leq \lim_{n\to\infty} \mathrm{tr}(v_n) = \mu_Q.$$

Now, let $\{w_n\}$ be an increasing sequence of moderate solutions such that $F_n := \mathfrak{T}_q$-supp $(w_n) \subset^q Q$ and $w_n \uparrow u_Q$. We must show that, if $\nu_n := \mathrm{tr}(w_n)$, then

$$\nu := \lim_{n\to\infty} \nu_n = \mu_Q. \tag{6.15}$$

By the previous argument $\nu \leq \mu_Q$. The opposite inequality is obtained as follows. Let D be a $\mathfrak{T}_q$-open set such that $[u]_D$ is moderate. Also, let K be a compact subset of D such that $C_{\frac{2}{q},\,q'}(K) > 0$.

$$w_n \leq [w_n]_D + [w_n]_{D^c} \Rightarrow u_Q = \lim_{n\to\infty} w_n \leq \lim_{n\to\infty} [w_n]_D + U_{D^c}.$$

The sequence $\{[w_n]_D\}$ is dominated by the moderate solution $[u_Q]_D$. In addition $\mathrm{tr}([w_n]_D) = \nu_n \chi_{\widetilde{D}} \uparrow \nu \chi_{\widetilde{D}}$. Hence, $\nu\chi_{\widetilde{D}}$ is a Radon measure which vanishes on sets with $C_{\frac{2}{q},\,q'}$-capacity zero. Also, $[w_n]_D \uparrow u_{\nu\chi_{\widetilde{D}}}$, where $u_{\nu\chi_{\widetilde{D}}}$ is a moderate solution with initial trace $\nu\chi_{\widetilde{D}}$. Consequently

$$u_Q = \lim_{n\to\infty} w_n \leq u_{\nu\chi_{\widetilde{D}}} + U_{D^c}.$$

This in turn implies

$$\Big([u_Q]_K - u_{\nu\chi_{\widetilde{D}}}\Big)_+ \leq \inf(U_{D^c}, U_K),$$

the function on the left being a subsolution and the one on the right a supersolution. Therefore

$$\Big([u_Q]_K - u_{\nu\chi_{\widetilde{D}}}\Big)_+ \leq [[U]_{D^c}]_K = 0.$$

Thus, $[u_Q]_K \leq u_{\nu\chi_{\widetilde{D}}}$ and hence $\mu_Q\chi_K \leq \nu\chi_{\widetilde{D}}$. Further, if O is a $\mathfrak{T}_q$-open set such that $\widetilde{O} \subset^q D$ then, in view of the fact that

$$\sup\{\mu_Q\chi_K :\ K \subset O,\ K \text{ compact}\} = \mu_Q\chi_O,$$

we obtain,

$$\mu_Q\chi_O \leq \nu\chi_{\widetilde{D}}. \tag{6.16}$$

Applying this inequality to the sets $Q_m,\ Q_{m+1}$ we finally obtain

$$\mu_Q\chi_{Q_m} \leq \nu\chi_{\widetilde{Q}_{m+1}} \leq \nu\chi_{Q_{m+2}}.$$

Letting $m \to \infty$ we conclude that $\mu_{\mathcal{R}_q} \leq \nu$. This completes the proof of (6.15). $\square$

6.2 $\mathfrak{T}_q$-perfect measures

Definition 6.6 *Let μ be a positive Borel measure on $\mathbb{R}^N$.*
(i) We say that μ is ***essentially absolutely continuous relative to*** $C_{\frac{2}{q},\,q'}$ *if the following condition holds:*

If Q is a $\mathfrak{T}_q$-open set and A is a Borel set such that $C_{\frac{2}{q},\,q'}(A) = 0$ then

$$\mu(Q \setminus A) = \mu(Q).$$

This relation be denoted by

$$\mu \prec\prec_f C_{\frac{2}{q},\,q'}.$$

(ii) μ is ***regular relative to $\mathfrak{T}_q$-topology*** *if, for every Borel set E,*

$$\begin{aligned} \mu(E) &= \inf\{\mu(D) : \ E \subset D, \ D\ \mathfrak{T}_q\text{−open}\} \\ &= \inf\{\mu(K) : \ K \subset E, \ K \text{ compact}\}. \end{aligned} \tag{6.17}$$

μ is ***outer regular relative to $\mathfrak{T}_q$-topology*** *if the first equality in (6.17) holds.*
*(iii) A positive Borel measure is called $\mathfrak{T}_q$**-perfect** if it is essentially absolutely continuous relative to $C_{\frac{2}{q},\,q'}$ and outer regular relative to $\mathfrak{T}_q$-topology. The space of $\mathfrak{T}_q$-perfect Borel measures is denoted by $\mathfrak{M}_q(\mathbb{R}^N)$.*

Proposition 6.7 *If $\mu \in \mathfrak{M}_q(\mathbb{R}^N)$ and A is a non-empty Borel set such that $C_{\frac{2}{q},\,q'}(A) = 0$, then*

$$\mu = \begin{cases} \infty & \text{if } \mu(Q \setminus A) = \infty \quad \forall Q\ \mathfrak{T}_q\text{-open neighborhood of } A, \\ 0 & \text{otherwise.} \end{cases} \tag{6.18}$$

If μ_0 is an essentially absolutely continuous positive measure on $\mathbb{R}^N$ and Q is $\mathfrak{T}_q$-open set such that $\mu_0(Q) < \infty$ then $\mu_0|_Q$ is ***absolutely continuous with respect to $C_{\frac{2}{q},\,q'}$ in the strong sense****, i.e., if $\{A_n\}$ is a sequence of Borel subsets of $\mathbb{R}^N$*

$$C_{\frac{2}{q},\,q'}(A_n) \to 0 \Rightarrow \mu_0(Q \cap A_n) \to 0.$$

Let μ_0 is an essentially absolutely continuous positive Borel measure on $\mathbb{R}^N$ and denote

$$\mu(E) = \inf\{\mu_0(D) : \ E \subset D, \ D\ \mathfrak{T}_q\text{-open}\}, \tag{6.19}$$

for every Borel set E; then

$$\begin{aligned} &(a) \quad \mu_0 \le \mu \qquad \mu_0(Q) = \mu(Q) \qquad \forall Q\ \ \mathfrak{T}_q\text{-open}, \\ &(b) \quad \mu|_Q = \mu_0|_Q \quad \text{for every } \mathfrak{T}_q\text{-open set } Q \ \text{ such that } \mu_0(Q) < \infty. \end{aligned} \tag{6.20}$$

Finally μ is $\mathfrak{T}_q$-perfect; thus μ is the smallest measure in $\mathfrak{M}_q(\mathbb{R}^N)$ which dominates μ_0.

Proof. The first assertion follows immediately from the definition $\mathfrak{M}_q(\mathbb{R}^N)$. We turn to the second assertion. If μ_0 is an essentially absolutely continuous positive Borel measure on $\mathbb{R}^N$, and Q is a $\mathfrak{T}_q$-open set such that $\mu_0(Q) < \infty$ then $\mu_0\chi_Q$ is a bounded Borel measure which vanishes on sets of $C_{\frac{2}{q},\,q'}-$ capacity zero. If $\{A_n\}$ is a sequence of Borel sets such that $C_{\frac{2}{q},\,q'}(A_n) \to 0$ and $\mu_n = \chi_{Q\cap A_n}$, then by Lemma 2.8-[19], there exists a unique moderate solution u_{μ_n}. Also in view of Lemma 2.8-[19] we can prove that the sequence $\{u_{\mu_n}\}$ is decreasing. Also by Proposition 5.17, we have $u_{\mu_n} \le U_{Q\cap A_n} \to 0$, since $C_{\frac{2}{q},\,q'}(Q\cap A_n) \to 0$. Thus we have that $u_{\mu_n} \to 0$ locally uniformly and $\mu_n \rightharpoonup 0$ weakly with respect to $C_0(\mathbb{R}^N)$. Hence $\mu(Q\cap A_n) \to 0$. Thus $\mu_0|_Q$ is absolutely continuous with respect to $C_{\frac{2}{q},\,q'}$ in the strong sense.

Assertion (6.20)(a) follows from (6.19). It is clear that μ, as defined by (6.19), is a measure. Now if Q is $\mathfrak{T}_q$-open set such that $\mu_0(Q) < \infty$, then $\mu(Q) < \infty$ and both $\mu_0|_Q$ and $\mu|_Q$ are regular. Since they agree on open sets, the regularity implies (6.20) (b).

If A is a Borel set such that $C_{\frac{2}{q},\,q'}(A) = 0$ and Q is a $\mathfrak{T}_q$-open set then $Q\setminus A$ is $\mathfrak{T}_q$-open and consequently

$$\mu(Q) = \mu_0(Q) = \mu_0(Q\setminus A) = \mu(Q\setminus A).$$

Thus μ is essentially absolutely continuous. By (6.20) (a) and the definition of μ, we have that μ is outer regular with respect to $C_{\frac{2}{q},\,q'}$. Thus $\mu \in \mathfrak{M}_q(\mathbb{R}^N)$. $\square$

6.3 The initial trace on the regular set

Proposition 6.8 *Let $u \in \mathcal{U}_+(Q_T)$.*
(i) There exists an increasing sequence of $\mathfrak{T}_q$-open sets $\{Q_n\}$ with the properties $Q_n \subset \mathcal{R}_q(u)$, $\widetilde{Q}_n \subset^q Q_{n+1}$ and $\mathcal{R}_{q,0}(u) := \bigcup_{n=1}^{\infty} Q_n \sim^q \mathcal{R}(u)$, such that the solution

$$v_n = u \wedge U_{Q_n} \text{ is moderate} \qquad v_n \uparrow v_{\mathcal{R}_q}, \qquad \mathrm{tr}(v_n) \to \mu_{\mathcal{R}_q}. \tag{6.21}$$

(ii)

$$v_{\mathcal{R}_q} := \sup\{[u]_F :\ F \subset^q \mathcal{R}_q(u),\ F\ \mathfrak{T}_q\text{−closed}\}. \tag{6.22}$$

Thus $v_{\mathcal{R}_q}$ is σ-moderate.
(iii) If $[u]_F$ is moderate and $F \subset^q \mathcal{R}_q(u)$, there exists a $\mathfrak{T}_q$-open set Q such that $F \subset^q Q$, $[u]_Q$ is moderate solution and $Q \subset \mathcal{R}_q(u)$
(iv) For every $\mathfrak{T}_q$-open set Q, such that $[u]_Q$ is a moderate solution, we have

$$\mu_{\mathcal{R}_q}\chi_{\widetilde{Q}} = \mathrm{tr}([u]_Q) = \mathrm{tr}([v_{\mathcal{R}_q}]_Q). \tag{6.23}$$

Finally, $\mu_{\mathcal{R}_q}$ is $\mathfrak{T}_q$-locally finite on $\mathcal{R}_q(u)$ and σ-finite on $\mathcal{R}_{q,0}(u) := \cup Q_n$.
(v) If $\{w_n\}$ is a sequence of moderate solutions such that $w_n \uparrow u_{\mathcal{R}_q}$ then,

$$\mu_{\mathcal{R}_q} = \lim_{n\to\infty} \mathrm{tr}(w_n) := \lim_{n\to\infty} \nu_n. \tag{6.24}$$

(vi) The regularized measure $\overline{\mu}_{\mathcal{R}_q}$ given by

$$\overline{\mu}_{\mathcal{R}_q}(E) = \inf\{\mu_{\mathcal{R}_q}(Q) :\ E \subset Q, \quad Q\ \mathfrak{T}_q\text{-open}, \quad E\ \ \textit{Borel}\}, \tag{6.25}$$

is $\mathfrak{T}_q$-perfect.
(vii)

$$u \approx_{\mathcal{R}_q(u)} v_{\mathcal{R}_q}.$$

(viii) For every $\mathfrak{T}_q$-closed set $F \subset^q \mathcal{R}_q(u)$:

$$[u]_F = [v_{\mathcal{R}_q}]_F. \tag{6.26}$$

If, in addition, $\mu_{\mathcal{R}_q}(F \cap K) < \infty$ for any compact $K \subset \mathbb{R}^N$, then $[u]_F$ is moderate and

$$\mathrm{tr}([u]_F) = \mu_{\mathcal{R}_q}\chi_F. \tag{6.27}$$

(ix) If F is a $\mathfrak{T}_q$-closed set and $C_{\frac{2}{q},\,q'}(F) > 0$ then

$$\mu_{\mathcal{R}_q}(F \cap K) < \infty \qquad \text{for any compact } K \subset \mathbb{R}^N \Leftrightarrow [u]_F \text{ is moderate.} \tag{6.28}$$

Proof. (i) For any $z \in \mathcal{R}_q(u)$ there exist a bounded $\mathfrak{T}_q$-open set $Q \subset \mathcal{R}_q(u)$ such that

$$\int_0^T \int_{\mathbb{R}^N} u^q(t,x)\mathbb{H}^{2q'}[\chi_Q]dxdt < \infty.$$

The result follows by similar arguments as in Lemma 6.4. Also, we recall that for any $z \in \mathcal{R}_q(u)$ there exists a $\mathfrak{T}_q$-open set $O_z \subset \mathcal{R}_q(u)$ such that

$$[u]_{O_z}, \tag{6.29}$$

is moderate.

Also we recall that $v_n = [v_{n+k}]_{Q_n}$, $\forall k \in \mathbb{N}$ and

$$\mu_n(Q_n) = \mu_{n+k}(Q_n) = \mu_{\mathcal{R}_q}(Q_n). \tag{6.30}$$

(ii) The proof is same as the one of Lemma 6.5-a)
(iii) First we assume that F is bounded. By definition and (6.29), every point in $\mathcal{R}_q(u)$ possesses a $\mathfrak{T}_q$-open neighborhood A such that $[u]_A$ is moderate. Then by Proposition 2.9, for any $\varepsilon > 0$ there exists a $\mathfrak{T}_q$-open set Q_ε such that $C_{\frac{2}{q},\,q'}(F \setminus Q_\varepsilon) < \varepsilon$ and $[u]_{Q_\varepsilon}$ is moderate. Since F is bounded, we can assume that so is Q_ε. Let O_ε be an open set containing $F \setminus Q_\varepsilon$ such that $C_{\frac{2}{q},\,q'}(O_\varepsilon) < 2\varepsilon$. Put

$$F_\varepsilon := F \setminus O_\varepsilon. \tag{6.31}$$

Then F_ε is a $\mathfrak{T}_q$-closed set, $F_\varepsilon \subset F$, $C_{\frac{2}{q},\,q'}(F \setminus F_\varepsilon) < 2\varepsilon$ and $F_\varepsilon \subset Q_\varepsilon$.
Assertion 1. Let E be a $\mathfrak{T}_q$-closed set, D a $\mathfrak{T}_q$-open set such that $[u]_D$ is moderate and $E \subset^q D$. Then there exists a decreasing sequence of $\mathfrak{T}_q$-open sets $\{G_n\}$ such that

$$E \subset^q G_{n+1} \subset \widetilde{G}_{n+1} \subset^q G_n \subset^q D, \tag{6.32}$$

and

$$[u]_{G_n} \to [u]_E \qquad \text{in } L^q(K) \text{ for any compact } K \subset \overline{Q}_T. \tag{6.33}$$

By Lemma 2.8 and Proposition 5.24-(iii), there exists a decreasing sequence of $\mathfrak{T}_q$-open sets $\{G_n\}$ satisfying (6.32) and, in addition, such that $[u]_{G_n} \downarrow [u]_E$ locally uniformly in $\mathbb{R}^N$. Since $[u]_{G_n} \leq [u]_D$ and the later is a moderate solution we obtain (6.33).

Now we assume that F is $\mathfrak{T}_q$-closed set (possibly unbounded). Let $x \in F$ and $B_n = B_n(x)$; $n \in \mathbb{N}$. Set

$$E_n = \bigcup_{m=1}^{n} (F \cap B_n)_{\frac{1}{2^m}},$$

where $(F \cap B_n)_{\frac{1}{2^m}}$ is the set in (6.31), if we replace F by $F \cap B_n$ and ε by $\frac{1}{2^m}$. Also we assume without loss of generality that $\{E_n\}$ is an increasing sequence. Also set

$$Q_n = \bigcup_{m=1}^{n} Q^n_{\frac{1}{m}},$$

where $Q^n_{\frac{1}{m}} = (F \cap B_n)_{\frac{1}{m}}$. Also we may assume that the sequence of set $\{Q_n\}$ is increasing. Therefore, we have that $E_n \subset E$, Q_n is $\mathfrak{T}_q$-open, $[u]_{Q_n}$ is moderate and $E_n \subset^q Q_n$ and $\cup E_n = E' \sim^q F$, since

$$C_{\frac{2}{q},\,q'}(F \setminus \bigcup_{n=1}^{\infty} E_n) \leq \sum_{k=1}^{n} C_{\frac{2}{q},\,q'}\left((F \cap B_k) \setminus \bigcup_{j=1}^{\infty} E_j\right) + \sum_{k=n+1}^{\infty} C_{\frac{2}{q},\,q'}\left((F \cap B_k) \setminus E_k\right)$$
$$\leq \frac{1}{2^n} + \sum_{k=n+1}^{\infty} \frac{1}{2^k}, \quad \forall n \in \mathbb{N}.$$

Thus by Assertion 1, it is possible to choose a sequence of $\mathfrak{T}_q$-open sets $\{V_n\}$ such that

$$E_n \subset^q V_n \subset \widetilde{V}_n \subset^q Q_n, \qquad ||[u]_{V_n} - [u]_{E_n}||_{L^q(B_n(0))\times(0,T]} \leq 2^{-n}. \tag{6.34}$$

We note here that since E_n, Q_n are bounded sets, the function $[u]_{V_n}$, $[u]_{E_n}$ have compact support with respect to variable "x" in $\mathbb{R}^N$, thus we can take the norm in (6.34) in whole space $\mathbb{R}^N \times (0,T]$.

Because $[u]_F$ is moderate, there exists a Radon measure μ_F where $\mu_F = \mathrm{tr}([u]_F)$. Moreover, $[u]_F = [u]_{E'}$ since $F \sim^q E'$. Finally, we have by (5.35) and the fact that $E_n \subset^q F$,

$$[u]_{E_n} = [u]_{E_n \cap F} = [[u]_{E_n}]_F.$$

Using the above equality and the fact that $[u]_F$ is moderate, we have that $\mathrm{tr}([u]_{E_n}) = \chi_{E_n}\mu_F$. Now since $E_n \uparrow E' \sim^q F$, it implies that $[u]_{E_n} \uparrow [u]_F$ $L^q(K \times [0,T])$, for compact set $K \subset \mathbb{R}^N$. Hence, we derive from by (6.34) that $[u]_{V_n} \to [u]_F$ in $L^q(K \times [0,T])$ for each compact set

$K \subset \mathbb{R}^N$.
Let $\{V_{n_k}\}$ be a sequence such that

$$\left(\int_0^1 \int_{B_k(0)} |[u]_{V_{n_k}} - [u]_F|^q dxdt\right)^{\frac{1}{q}} \leq 2^{-k}. \tag{6.35}$$

If $K$is a compact set, there exist $k_0 \in \mathbb{N}$ such that $K \subset B_k(0),\ \forall k \geq k_0$. Set $W = \bigcup_{k=1}^{\infty} V_{n_k}$, then

$$[u]_W \leq \sum_{k=1}^{\infty} [u]_{V_{n_k}}.$$

Thus we have

$$\begin{aligned}
\left(\int_0^T \int_K |[u]_W - [u]_F|^q dxdt\right)^{\frac{1}{q}} &\leq \sum_{k=1}^{k_0} \left(\int_0^T \int_K |[u]_{V_{n_k}} - [u]_F|^q dxdt\right) \\
&\quad + \sum_{k=k_0+1}^{\infty} \left(\int_0^T \int_{B_k(0)} |[u]_{V_{n_k}} - [u]_F|^q dxdt\right)^{\frac{1}{q}} \\
&\leq \sum_{k=1}^{k_0} \left(\int_0^T \int_K |[u]_{V_{n_k}} - [u]_F|^q dxdt\right) + \sum_{k=k_0+1}^{\infty} 2^{-k} \\
&< \infty.
\end{aligned}$$

We recall that $F \subset^q W$ and W is a $\mathfrak{T}_q$-open set. Using the facts that $[u]_F$ is moderate, K is an abstract compact domain and the above inequality, we obtain that $[u]_W$ is moderate. Thus by Lemma 6.2 we have that $W \subset \mathcal{R}_q(u)$.
(iv) Let Q be a $\mathfrak{T}_q$-open set and $[u]_Q$ be a moderate solution, put $\mu_Q = \mathrm{tr}([u]_Q)$. If F is a $\mathfrak{T}_q$-closed set such that $F \subset^q Q$ then, by Proposition 5.24-(ii),

$$\mathrm{tr}[u]_F = \mathrm{tr}([[u]_Q]_F) = \mu_Q \chi_F. \tag{6.37}$$

In particular the compatibility condition holds: if $Q,\ Q'$ are $\mathfrak{T}_q$-open regular sets then

$$\mu_{Q\cap Q'} = \mu_Q \chi_{\widetilde{Q}\cap\widetilde{Q}'} = \mu_{Q'} \chi_{\widetilde{Q}\cap\widetilde{Q}'}. \tag{6.38}$$

With the notation of (i), $[v_{n+k}]_{Q_k} = v_k$ and hence $\mu_{n+k}\chi_{\widetilde{Q}_k} = \mu_k$ for every $k \in \mathbb{N}$.

Let F be an arbitrary $\mathfrak{T}_q$-closed regular subset of $\mathcal{R}_q(u)$. Since $[u]_F$ is moderate, we have by (6.38)

$$[v_n]_F = [u]_{F\cap\widetilde{Q}_n} \uparrow [u]_F. \tag{6.39}$$

In addition, $[v_{\mathcal{R}_q}]_F \geq \lim_{n\to\infty}[v_n]_F = [u]_F$, jointly with $v_{\mathcal{R}_q} \leq u$, leads to,

$$[u]_F = [v_{\mathcal{R}_q}]_F. \tag{6.40}$$

By (6.37) and (6.39), if F is a $\mathfrak{T}_q$-closed subset of $\mathcal{R}_q(u)$ and $[u]_F$ is moderate,

$$\mathrm{tr}([u]_F) = \lim_{n\to\infty} \mathrm{tr}([v_n]_F) = \lim_{n\to\infty} \mu_n \chi_F = \mu_{\mathcal{R}_q}\chi_F, \tag{6.41}$$

which implies (6.23).

Since $\mathcal{R}_{q,0}(u)$ has a regular decomposition, $\mu_{\mathcal{R}_q}$ is σ-finite on $\mathcal{R}_{q,0}(u)$. The assertion that $\mu_{\mathcal{R}_q}$ is $\mathfrak{T}_q$-locally finite on $\mathcal{R}_q(u)$ is a consequence of the fact that every point $\xi \in \mathcal{R}_q(u)$ is contained in a $\mathfrak{T}_q$-open set $O_\xi \subset \mathcal{R}_q(u)$ such that $[u]_{O_\xi}$ is moderate and thus $\mu_{\mathcal{R}_q}\chi_{O_\xi} < \infty$.
(v) If w is a moderate solution and $w \leq v_{\mathcal{R}_q}$ and $\mathfrak{T}_q$-supp $(w) \subset^q \mathcal{R}_q(u)$ then $\tau := \mathrm{tr}(w) \leq \mu_{\mathcal{R}_q}$. Indeed

$$[w]_{Q_n} \leq [v_{\mathcal{R}_q}] = v_n,\ [w]_{Q_n} \uparrow w \Rightarrow \mathrm{tr}([w]_{Q_n}) \uparrow \tau \leq \lim_{n\to\infty} \mathrm{tr}(v_n) = \mu_{\mathcal{R}_q}.$$

Now, let $\{w_n\}$ be an increasing sequence of moderate solutions such that $F_n := \mathfrak{T}_q$-supp $(w_n) \subset^q \mathcal{R}_q(u)$ and $w_n \uparrow v_{\mathcal{R}_q}$. If $\nu_n := \mathrm{tr}(w_n)$, we have to prove that

$$\nu := \lim_{n\to\infty} \nu_n = \mu_{\mathcal{R}_q}. \tag{6.42}$$

By the previous argument $\nu \leq \mu_{\mathcal{R}_q}$. The opposite inequality is obtained as follows. Let D be a $\mathfrak{T}_q$-open set, $[u]_D$ be moderate and let K be a compact subset of D such that $C_{\frac{2}{q},\,q'}(K) > 0$.

$$w_n \leq [w_n]_D + [w_n]_{D^c} \Rightarrow v_{\mathcal{R}_q} = \lim_{n\to\infty} w_n \leq \lim_{n\to\infty} [w_n]_D + U_{D^c}.$$

The sequence $\{[w_n]_D\}$ is dominated by the moderate function $[v_{\mathcal{R}_q}]_D$. In addition $\mathrm{tr}([w_n]_D) = \nu_n\chi_{\widetilde{D}} \uparrow \nu\chi_{\widetilde{D}}$. Hence, $\nu\chi_{\widetilde{D}}$ is a Radon measure which vanishes on sets with $C_{\frac{2}{q},\,q'}$-capacity zero. Also, $[w_n]_D \uparrow u_{\nu\chi_{\widetilde{D}}}$, where the function $u_{\nu\chi_{\widetilde{D}}}$ on the right is the moderate solution with initial trace $\nu\chi_{\widetilde{D}}$. Consequently

$$v_{\mathcal{R}_q} = \lim_{n\to\infty} w_n \leq u_{\nu\chi_{\widetilde{D}}} + U_{D^c}.$$

This in turn implies

$$\Big([v_{\mathcal{R}_q}]_K - u_{\nu\chi_{\widetilde{D}}}\Big)_+ \leq \inf(U_{D^c}, U_K).$$

Note that, in the previous relation, the function on the left being a subsolution and the one on the right a supersolution, we obtain

$$\Big([v_{\mathcal{R}_q}]_K - u_{\nu\chi_{\widetilde{D}}}\Big)_+ \leq [[U]_{D^c}]_K = 0.$$

Thus, $[v_{\mathcal{R}_q}]_K \leq u_{\nu\chi_{\widetilde{D}}}$ and hence $\mu_R\chi_K \leq \nu\chi_{\widetilde{D}}$. Further, if Q is a $\mathfrak{T}_q$-open set such that $\widetilde{Q} \subset^q D$ then, in view of the fact that

$$\sup\{\mu_{\mathcal{R}_q}\chi_K:\ K \subset Q,\ K \text{ compact}\} = \mu_{\mathcal{R}_q}\chi_Q,$$

we obtain,

$$\mu_{\mathcal{R}_q}\chi_Q \leq \nu\chi_{\widetilde{D}}. \tag{6.43}$$

Applying this inequality to the sets $Q_m,\ Q_{m+1}$ we finally obtain

$$\mu_{\mathcal{R}_q}\chi_{Q_m} \leq \nu\chi_{\widetilde{Q}_{m+1}} \leq \nu\chi_{Q_{m+2}}.$$

Letting $m \to \infty$ we conclude that $\mu_{\mathcal{R}_q} \leq \nu$. This completes the proof of (6.42) and of assertion (v).

(vi) The measure $\mu_{\mathcal{R}_q}$ is essentially absolutely continuous relative to $C_{\frac{2}{q},\,q'}$. Clearly this assertion follows now from Proposition 6.7.
(vii) By (5.34)

$$u \leq [u]_{Q_n} + [u]_{Q_n^c}.$$

Now since Q_n^c is $\mathfrak{T}_q$-closed and $Q_n^c \downarrow \mathcal{R}_{q,0}^c(u)$, we have by Proposition 5.24-(iii) that

$$[u]_{Q_n^c} \downarrow [u]_{\mathcal{R}_{q,0}^c(u)}.$$

Hence

$$\lim_{n\to\infty} (u - [u]_{Q_n}) = u - v_{\mathcal{R}_q} \leq [u]_{\mathcal{R}_{q,0}^c(u)},$$

therefore $u \ominus v_{\mathcal{R}_q} \approx_{\mathcal{R}_{q,0}(u)} 0$. Since $v_{\mathcal{R}_q} \leq u$ this is equivalent to the statement $u \approx_{\mathcal{R}_{q,0}(u)} v_{\mathcal{R}_q}$.
(viii) (6.26) follows by the previous statement. Now we assume that $\mu_{\mathcal{R}_q}(F)\chi_K < \infty$ for any compact $K \subset \mathbb{R}^N$. Now set $F_n = F \cap \widetilde{Q}_n$. By (5.34).

$$[u]_F \leq [u]_{F_n} + [u]_{F\setminus F_n} = [u]_{F_n} + [u]_{F\setminus \widetilde{Q}_n} \leq [u]_{F_n} + [u]_{F\setminus Q_n}.$$

Now since $F \setminus Q_n$ is a $\mathfrak{T}_q$-closed set and $\cap F \setminus Q_n = G$ with $C_{\frac{2}{q},\,q'}(G) = 0$, we have by Proposition 5.24-(iii) that $[u]_{F\setminus Q_n} \to [u]_G = 0$. Hence $[u]_F = \lim_{n\to\infty}[u]_{F_n}$, and $\mathrm{tr}([u]_{F_n}) = \mu_{\mathcal{R}_q}\chi_{F_n} \uparrow \mu_{\mathcal{R}_q}\chi_{F_0} = \mu_{\mathcal{R}_q}\chi_F$. Since $\mu_R\chi_F$ is a Radon measure essentially absolutely continuous relative to $C_{\frac{2}{q},\,q'}$, $[u]_F$ is moderate and (6.27) holds.
(ix) If $\mu_{\mathcal{R}_q}(F)\chi_K < \infty$ for any compact $K \subset \mathbb{R}^N$ then, by (viii), $[u]_F$ is moderate. Conversely, if $[u]_F$ is moderate, by (6.23), $\mu_{\mathcal{R}_q}(F)\chi_K < \infty$ for any compact $K \subset \mathbb{R}^N$. □

Example. We give below an example which shows that there exists $u \in \mathcal{U}_+(Q_T)$ such that $\mathcal{R}_q(u) = \mathbb{R}^N$ but u is not a moderate solution. Let $\eta : [0,\infty) \to [0,\infty)$ be a smooth function such that $\eta(r) > 0$ for any $r > 0$ and $\lim_{r\to 0^+} \eta(r) = 0$, ($\eta$ tends to 0 very fast, for example $\eta(r) = e^{-\frac{1}{r^2}}$). Let K be the close set

$$K = \{(x', x_n) \in \mathbb{R}^N : |x'| \leq \eta(x_n),\ x_n \geq 0\}.$$

Then K is thin at the origin 0.

Set $f(x) = \frac{1}{\eta^n(x_n)}$ if $x \in K$ and $f = 0$ otherwise. We define the measure

$$\mu = f dx.$$

This measure possesses the following properties:
1. μ is $\mathfrak{T}_q$-locally finite.
2. $\mu(Q_n) < \infty$ where $Q_n = B_{2n}(0) \setminus \overline{B}_{\frac{1}{n}}(0)$ and $\cup Q_n \sim^q \mathbb{R}^N$
3. $\mu(F) = 0$ for any F such that $C_{\frac{2}{q},\,q'}(F) = 0$.
4. There exists a non decreasing sequence of bounded Radon measures μ_n absolutely continuous with $C_{\frac{2}{q},\,q'}$ such that
(a) $\mathfrak{T}_q$-supp $(\mu_n) \subset \widetilde{Q}_n$, $\mu_n(A) = \mu_{n+k}(A)$ for any $A \subset \widetilde{Q}_n$ and any $n, k \in \mathbb{N}$.
(b) $\lim_{n\to\infty} \mu_n = \mu$
5. We can construct a solution $u \in \mathcal{U}_+(Q_T)$ with respect to this measure.
As we see later this solution is unique since it is σ-moderate (see Proposition 6.12).

Lemma 6.9 *Let μ satisfying the conditions 1-4 as above. Then there exists an open set $\mathcal{R}_q \sim^q \mathbb{R}^N$ such that the measure μ is a Radon measure in $\mathcal{R}_q$.*

Proof. We consider the ball $B_R(0)$ with $R > 1$. From [23, Lemma 2.5] there exists a sequence of open sets $\{O_{\frac{1}{m}}\}_{m=1}^{\infty}$ and $n(m) \in \mathbb{N}$ such that $C_{\frac{2}{q},q'}(O_{\frac{1}{m}}) < \frac{1}{m}$, and

$$\overline{B}_R(0) \setminus O_{\frac{1}{m}} \subset \bigcup_{i=i}^{n(m)} Q_i. \tag{6.44}$$

Now since $O_{\frac{1}{m}}$ is open we have

$$C_{\frac{2}{q},q'}(\overline{O}_{\frac{1}{m}}) = C_{\frac{2}{q},q'}(\widetilde{O}_{\frac{1}{m}} \bigcup (\overline{O}_{\frac{1}{m}} \cap e_q(O))) \leq C_{\frac{2}{q},q'}(\widetilde{O}_{\frac{1}{m}}) \leq cC_{\frac{2}{q},q'}(O_{\frac{1}{m}}) \to 0,$$

where $e_q(O)$ is the set of thin points of O.
Thus if $x \in B_R(0) \setminus \bigcap_{m=1}^{\infty} \overline{O}_{\frac{1}{m}}$ there exist $r > 0$ small enough and $N \in \mathbb{N}$ such that

$$B_r(x) \subset B_R(0) \setminus \bigcap_{m=1}^{N} \overline{O}_{\frac{1}{m}}.$$

Thus by the properties of μ and (6.44) we have

$$\mu(B_r(x)) < \infty.$$

We define

$$\mathcal{R}_q := \{x \in \mathbb{R}^N : \exists\, r > 0 \text{ such that } \mu(B_r(x)) < \infty\}.$$

Then the set $\mathcal{R}_q$ is open and by the above argument, letting R go to infinity, we have that $\mathcal{R}_q \sim^q \mathbb{R}^N$. Also by the definition of $\mathcal{R}_q$, it is easy to see that $\mu(K) < \infty$ for any compact $K \subset \mathcal{R}_q$ and by the properties of μ we can prove that μ is Radon measure in $\mathcal{R}_q$. $\square$

6.4 The precise initial trace

We are now in condition to define the *precise initial trace.*

Definition 6.10 *Let $q \geq 1 + \frac{2}{N}$ and $u \in \mathcal{U}_+(Q_T)$.*
a: *The solution $v_{\mathcal{R}_q}$ defined by (6.22) is called **regular component of** u and will be denoted by u_{reg}.*
***b**: Let $\{v_n\}$ be an increasing sequence of moderate solutions satisfying condition (6.21) and put $\mu_{\mathcal{R}_q} = \mu_{\mathcal{R}_q}(u) := \lim_{n\to\infty} \operatorname{tr}(v_n)$. Then, the regularized measure $\overline{\mu}_{\mathcal{R}_q}$, defined by (6.25), is called the **regular initial trace of** u. It will be denoted by $\operatorname{tr}_{\mathcal{R}_q}(u)$.*
c: *The couple $(\operatorname{tr}_{\mathcal{R}_q}(u), \mathcal{S}_q(u))$ is called **the precise initial trace of** u and will be denoted by $\operatorname{tr}^c(u)$.*
d: *Let ν be the Borel measure on $\mathbb{R}^N$ given by*

$$\nu = \begin{cases} (\operatorname{tr}_{\mathcal{R}_q}(u))(E) & \text{if } E \subset \mathcal{R}_q(u), \\ \nu(E) = \infty & \text{if } E \cap \mathcal{S}_q(u) \neq \emptyset, \end{cases} \tag{6.45}$$

for every Borel set E. Then ν is the measure representation of the precise trace of u, to be denoted by $\operatorname{tr}(u)$.

Note that, by Proposition 6.8-(v), the measure $\mu_{\mathcal{R}_q}$ is independent of the choice of the sequence $\{v_n\}$.

Theorem 6.11 *Assume that $u \in \mathcal{U}_+(Q_T)$ is a σ-moderate solution, i.e., there exists an increasing sequence $\{u_n\}$ of positive moderate solutions such that $u_n \uparrow u$. Let $\mu_n = \lim_{n\to\infty} \operatorname{tr}(u_n)$, $\mu_0 := \lim_{n\to\infty} \mu_n$ and set, for any Borel set E,*

$$\mu(E) = \inf\{\mu_0(Q) : \ E \subset Q, \quad Q \ \mathfrak{T}_q\text{-}open\}. \tag{6.46}$$

Then:
(i) μ is the precise initial trace of u and μ is $\mathfrak{T}_q$-perfect. In particular μ is independent of the sequence $\{u_n\}$ which appears in its definition.
(ii) If A is a Borel set such that $\mu(A) < \infty$, then $\mu(A) = \mu_0(A)$.
(iii) A solution $u \in \mathcal{U}_+(Q_T)$ is σ-moderate if and only if

$$u = \sup\{v \in \mathcal{U}_+(Q_T) : \ v \text{ moderate}, \quad v \le u\}, \tag{6.47}$$

which is equivalent to

$$u = \sup\{u_\tau \in \mathcal{U}_+(Q_T) : \ \tau \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N), \quad \tau \le \operatorname{tr}(u)\}. \tag{6.48}$$

(iv) If $u, \ w$ are σ-moderate solutions,

$$\operatorname{tr}(w) \le \operatorname{tr}(u) \Leftrightarrow w \le u. \tag{6.49}$$

Proof. The proof is an adaptation of the one in [23].
(i) Let Q be a $\mathfrak{T}_q$-open set and A a Borel set such that $C_{\frac{2}{q},\,q'}(A) = 0$. Then $\mu_n(A) = 0$ so that $\mu_0(A) = 0$. Thus μ_0 is essentially absolutely continuous and, by Proposition 6.7, μ is $\mathfrak{T}_q$-perfect.

Let $\{D_n\}$ be the family of $\mathfrak{T}_q$-open sets as in Proposition 6.8-(i). Put $D'_n = \mathcal{R}_q(u) \setminus D_n$ and observe that $D'_n \downarrow E$ where $C_{\frac{2}{q},\,q'}(E) = 0$. Therefore, for fixed n,

$$u_{\mu_n \chi_{D'_m}} \downarrow 0 \quad \text{when } m \to \infty.$$

Thus there exist a subsequence, say $\{D'_n\}$, such that

$$\left(\int_0^T \int_{B_n(0)} |u_{\mu_n \chi_{D'_n}}|^q dx dt \right)^{\frac{1}{q}} \le 2^{-n}.$$

Since,

$$\mu_n(\mathcal{R}_q(u)) = \mu_n \chi_{D_n} + \mu_n \chi_{D'_n},$$

it follows that

$$\lim_{n\to\infty} \left| u_{\mu_n \chi_{\mathcal{R}_q(u)}} - u_{\mu_n \chi_{D_n}} \right| \le \lim_{n\to\infty} u_{\mu_n \chi_{D'_n}} = 0.$$

Thus

$$u_n \le u_{\mu_n \chi_{D_n}} + [u]_{\mathcal{S}_q(u)}.$$

Hence

$$u - [u]_{\mathcal{S}_q(u)} \le w := \lim_{n\to\infty} u_{\mu_n \chi_{\mathcal{R}_q(u)}} = \lim_{n\to\infty} u_{\mu_n \chi_{D_n}} \le u_{reg}.$$

This implies $u \ominus [u]_{\mathcal{S}_q(u)} \le u_{reg}$. For the opposite inequality, by Proposition 6.8-(iv) we get

$$[u]_{D_n} \uparrow u_{reg}.$$

But by (5.46) and using the facts that $\widetilde{D}_n \subset^q D_{n+1} \subset \widetilde{D}_{n+1} \subset^q \mathcal{R}_q(u)$, $C_{\frac{2}{q},q'}\left(\widetilde{D}_{n+1} \cap \mathcal{S}_q(u)\right) = 0$,

$$[u]_{D_n} \le [[u]_{\mathcal{S}_q(u)}]_{D_{n+1}} + [u \ominus [u]_{\mathcal{S}_q(u)}]_{D_{n+1}} = [u \ominus [u]_{\mathcal{S}_q(u)}]_{D_{n+1}} \le u \ominus [u]_{\mathcal{S}_q(u)}.$$

Letting $n \to \infty$ we derive $u_{reg} \le u \ominus [u]_{\mathcal{S}_q(u)}$. Therefore $\lim_{n\to\infty} u_{\mu_n \chi_{D_n}} = u_{reg}$. Thus the sequence $\{u_{\mu_n \chi_{D_n}}\}$ satisfies condition (6.21) and consequently, by Proposition 6.8-(iv) and Definition 6.10,

$$\lim_{n\to\infty} \mu_n \chi_{D_n} = \mu_{\mathcal{R}_q}, \quad \mathrm{tr}_{\mathcal{R}_q}(u) = \overline{\mu}_{\mathcal{R}_q}. \tag{6.50}$$

Next, we claim that, if $\xi \in \mathcal{S}_q(u)$ then, for every $\mathfrak{T}_q$-open bounded neighborhood Q of ξ $\mu_n(\widetilde{Q}) \to \infty$. Indeed let $\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\mathfrak{T}_q$-support in Q. Put $h = \mathbb{H}[\eta]$ and $\phi(r) = r_+^{2q'}$. Then by Proposition 5.7, Lemma 5.3 and in view of the proof of Lemma 5.2 we have

$$\int_0^T \int_{\mathbb{R}^N} (-u_n(\partial_t \phi(h) + \Delta\phi(h))) + u_n^q \phi(h) dx d\tau + \int_{\mathbb{R}^N} u_n \phi(h)(.,T) dx = \int_Q \eta^{2q'} d\mu_n.$$

In view of Lemma 5.2, we can prove

$$\int_0^T \int_{\mathbb{R}^N} u_n^q \phi(h) dx d\tau \le C(q) \left(\int_Q \eta^{2q'} d\mu_n + ||\eta||^{2q'}_{W^{\frac{2}{q},q'}} + ||\eta||_{L^\infty} \right).$$

By Lemma 4.2 there exist $\eta \in W^{\frac{2}{q},q'}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and a $\mathfrak{T}_q$-open set $D \subset Q$ such that $\eta = 1$ on D, $\eta = 0$ outside of Q and $0 \le \eta \le 1$. Letting $n \to \infty$ we have

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^N} u_n^q \mathbb{H}^{2q'}[\chi_D] dx d\tau \le C(q) \left(\lim_{n\to\infty} \int_Q \eta^{2q'} d\mu_n + ||\eta||^{2q'}_{W^{\frac{2}{q},q'}} + ||\eta||_{L^\infty} \right),$$

the assertion follows by Lemma 5.4.

In conclusion, if $\xi \in \mathcal{S}_q(u)$ then $\mu_0(\widetilde{Q}) = \infty$ for every $\mathfrak{T}_q$-open neighborhood of ξ. Consequently $\mu(\xi) = \infty$. This fact and (6.50) imply that μ is the precise trace of u.

(ii) If $\mu(A) < \infty$ then A is contained in a $\mathfrak{T}_q$-open set D such that $\mu_0(D) < \infty$ and, by Proposition 6.7, $\mu(A) = \mu_0(A)$.

(iii) Let $u \in \mathcal{U}_+(Q_T)$ be σ-moderate and put

$$u^* := \sup\{v : \ v \text{ moderate}, \quad v \le u\}. \tag{6.51}$$

By its definition $u^* \le u$. On the other hand, since there exists an increasing sequence of moderate solutions $\{u_n\}$ converging to u, it follows that $u \le u^*$. Thus $u = u^*$.

Conversely, if $u \in \mathcal{U}_+(Q_T)$ and $u = u^*$ then by proposition 3.4, there exists an increasing sequence of moderate solutions $\{u_n\}$ converging to u. Therefore u is σ-moderate.

Since u is σ-moderate there exist an increasing sequence of moderate solutions $\{u_n\}$ converging to u. In view of the discussion at the beginning of subsection 5.1, for any u_n there exist an increasing sequence of $\{w_m\}$ such that $w_m \uparrow u_n$ and $\mathrm{tr}(w_m) \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$. Thus

$$u_n \leq \sup\{u_\tau : \tau \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N), \quad \tau \leq \mathrm{tr}(u)\} =: u^\ddagger.$$

Letting $n \to \infty$, we have $u \leq u^\ddagger$.

On the other hand, if u is σ-moderate, $\tau \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$ and $\tau \leq \mathrm{tr}(u)$ then (with μ_n and u_n as in the statement of the Proposition), $\mathrm{tr}(u_\tau \ominus u_n) = (\tau - \mu_n)_+ \downarrow 0$. Hence, $u_\tau \ominus u_n \downarrow 0$, which implies, $u_\tau \leq u$. Therefore $u^\ddagger \leq u$. Thus (6.47) implies (6.48) and each of them that u is σ-moderate. Therefore the two statements are equivalent.
(iv) The assertion $\Rightarrow$ is a consequence of (6.48). To verify the assertion $\Leftarrow$ it is sufficient to show that if w is moderate, u is σ-moderate and $w \leq u$, then $\mathrm{tr}(w) \leq$ u. Let $\{u_n\}$ be an increasing sequence of positive moderate solutions converging to u. Then $u_n \vee w \leq u$ and consequently $u_n \leq u_n \vee w \uparrow u$. Therefore $\mathrm{tr}(u_n \vee w) \uparrow \mu' \leq \mathrm{tr}(u)$ so that $\mathrm{tr}(w) \leq \mathrm{tr}(u)$. $\square$

Theorem 6.12 *Let $u \in \mathcal{U}_+(Q_T)$ and put $\nu = \mathrm{tr}(u)$.*
(i) u_{reg} is σ-moderate and $\mathrm{tr}(u_{reg}) = \mathrm{tr}_{\mathcal{R}_q}(u)$.
(ii) If $v \in \mathcal{U}_+(Q_T)$

$$v \leq u \Rightarrow \mathrm{tr}(v) \leq \mathrm{tr}(u). \tag{6.52}$$

If F is a $\mathfrak{T}_q$-closed set, then

$$\mathrm{tr}([u]_F) \leq \nu \chi_F. \tag{6.53}$$

(iii) A singular point can be characterized in terms of the measure ν as follows:

$$\xi \in \mathcal{S}_q(u) \Leftrightarrow \nu(Q) = \infty \qquad \forall Q : \ \xi \in Q, \ Q \ \mathfrak{T}_q\textit{-open}. \tag{6.54}$$

(iv) If Q is a $\mathfrak{T}_q$-open set then:

$$[u]_Q \text{ is moderate} \quad \Leftrightarrow \quad \exists \text{ Borel set } A : \ C_{\frac{2}{q},\, q'}(A) = 0, \ \nu(K \cap \widetilde{Q} \setminus A) < \infty, \tag{6.55}$$

for any compact $K \subset \mathbb{R}^N$.
(v) The singular set of u_{reg} may not be empty. In fact

$$\mathcal{S}_q(u) \setminus b_q(\mathcal{S}_q(u)) \subset \mathcal{S}_q(u_{reg}) \subset \mathcal{S}_q(u) \cap \widetilde{\mathcal{R}_q(u)}, \tag{6.56}$$

where $b_q(\mathcal{S}_q(u))$ is the set of $C_{\frac{2}{q},\, q'}$-thick points of $\mathcal{S}_q(u)$.
(vi) Put

$$\mathcal{S}_{q,0}(u) := \{\xi \in \mathbb{R}^N : \ \nu(Q \setminus \mathcal{S}_q(u)) = \infty \quad \forall \ Q \ \ \mathfrak{T}_q\textit{-open neighborhood of } \xi\}. \tag{6.57}$$

Then

$$\mathcal{S}_q(u_{reg}) \setminus b_q(\mathcal{S}_q(u)) \subset \mathcal{S}_{q,0}(u) \subset \mathcal{S}_q(u_{reg}) \bigcup b_q(\mathcal{S}_q(u)). \tag{6.58}$$

Remark. This results extends Proposition 6.8 which deals with the regular initial trace.

Proof. (i) By proposition 6.8-(ii) u_{reg} is σ-moderate. The second part of the statement follows from Definition 6.10 and Proposition 6.11-(i).
(ii) If $v \leq u$ then $\mathcal{R}_q(u) \subset \mathcal{R}_q(v)$ and by definition $v_{reg} \leq u_{reg}$. By Proposition 6.11-(iv) $\operatorname{tr}(v_{reg}) \leq \operatorname{tr}(u_{reg})$ and consequently $\operatorname{tr}(v) \leq \operatorname{tr}(u)$. Inequality (6.53) is an immediate consequence of (6.52).
(iii) If $\xi \in \mathcal{R}_q(u)$ there exists a $\mathfrak{T}_q$-open neighborhood Q of ξ such that $[u]_Q$ is moderate. Hence $\nu(Q) = \operatorname{tr}_{\mathcal{R}_q}(u)(Q) < \infty$. If $\xi \in \mathcal{S}_q(u)$, it follows immediately from the definition of precise trace that $\nu(Q) = \infty$ for every $\mathfrak{T}_q$-open neighborhood Q of ξ.
(iv) If $[u]_Q$ is moderate then $Q \subset \mathcal{R}_q(u)$. Proposition 6.8-(ix) implies (6.55) in the direction $\Rightarrow$. On the other hand,

$$\nu(K \cap \widetilde{Q} \setminus A) < \infty, \; \forall \text{ compact } K \subset \mathbb{R}^N \Rightarrow \widetilde{Q} \subset^q \mathcal{R}_q(u),$$

and $\mu_{\mathcal{R}_q}(K \cap \widetilde{Q}) = \mu_{\mathcal{R}_q}(K \cap \widetilde{Q} \setminus A) < \infty$. Hence, by Proposition 6.8-(ix), $[u]_Q$ is moderate.
(v) Since $\mathfrak{T}_q$-supp $(u_{reg}) \subset \widetilde{\mathcal{R}_q(u)}$ and $\mathcal{R}_q(u) \subset \mathcal{R}_q(u_{reg})$ we have

$$\mathcal{S}_q(u_{reg}) \subset \mathcal{S}_q(u) \cap \widetilde{\mathcal{R}_q(u)}.$$

Next we show that $\mathcal{S}_q(u) \setminus b_q(\mathcal{S}_q(u)) \subset \mathcal{S}_q(u_{reg})$.

If $\xi \in \mathcal{S}_q(u) \setminus b_q(\mathcal{S}_q(u))$ then $\mathcal{R}_q(u) \cup \{\xi\}$ is a $\mathfrak{T}_q$-open neighborhood of ξ. By (i) u_{reg} is σ-moderate and consequently (by Proposition 6.11-(i)) its trace is $\mathfrak{T}_q$-perfect. Therefore, if Q_0 is a bounded $\mathfrak{T}_q$-open neighborhood of ξ and $Q = Q_0 \cap (\{\xi\} \cup \mathcal{R}_q(u))$ then

$$\operatorname{tr}(u_{reg})(Q) = \operatorname{tr}(u_{reg})(Q \setminus \{\xi\}) = \operatorname{tr}(u)(Q \setminus \{\xi\}),$$

where in the last equality we have used the fact that $Q \setminus \{\xi\} \subset \mathcal{R}_q(u)$. Let D be a $\mathfrak{T}_q$-open set such that $\xi \in D \subset \widetilde{D} \subset Q$. If $\operatorname{tr}(u)(\widetilde{D} \setminus \{\xi\}) < \infty$ then, by (iv), $[u]_D$ is moderate and $\xi \in \mathcal{R}_q(u)$, contrary to our assumption. Therefore $\operatorname{tr}(u)(\widetilde{D} \setminus \{\xi\}) = \infty$ so that $\operatorname{tr}(u_{reg})(Q_0 \setminus \{\xi\}) = \infty$ for every $\mathfrak{T}_q$-open bounded neighborhood Q_0 of ξ, which implies $\xi \in \mathcal{S}_q(u_{reg})$. This completes the proof of (6.56).
(vi) If $\xi \notin b_q(\mathcal{S}_q(u))$, there exists a $\mathfrak{T}_q$-open neighborhood D of ξ such that $(D \setminus \{\xi\}) \cap \mathcal{S}_q(u) = \emptyset$ and consequently

$$\operatorname{tr}(u_{reg})(D \setminus \{\xi\}) = \operatorname{tr}(u_{reg})(D \setminus \mathcal{S}_q(u)) = \operatorname{tr}(u)(D \setminus \mathcal{S}_q(u)). \tag{6.59}$$

If, in addition $\xi \in \mathcal{S}_{q,0}(u)$ then

$$\operatorname{tr}(u)(D \setminus \mathcal{S}_q(u)) = \operatorname{tr}(u_{reg})(D \setminus \mathcal{S}_q(u)) = \infty.$$

If Q is an arbitrary $\mathfrak{T}_q$-open neighborhood of ξ then the same holds if D is replaced by $Q \cap D$. Therefore $\operatorname{tr}(u_{reg})(Q \setminus \{\xi\}) = \infty$ for any such Q. Consequently $\xi \in \mathcal{S}_q(u_{reg})$, which proves that $\mathcal{S}_{q,0}(u) \setminus b_q(\mathcal{S}_q(u)) \subset \mathcal{S}_q(u_{reg})$.

On the other hand, if $\xi \in \mathcal{S}_q(u_{reg}) \setminus b_q(\mathcal{S}_q(u))$ then there exists a $\mathfrak{T}_q$-open neighborhood D such that (6.59) holds and $\operatorname{tr}(u_{reg})(D) = \infty$. Since u_{reg} is σ-moderate, $\operatorname{tr}(u_{reg})$ is $\mathfrak{T}_q$-perfect so that $\operatorname{tr}(u_{reg})(D) = \operatorname{tr}(u_{reg})(D \setminus \{\xi\}) = \infty$. Consequently, by (6.59), $\operatorname{tr}(u)(D \setminus \mathcal{S}_q(u)) = \infty$. If Q is any $\mathfrak{T}_q$-open neighborhood of $\{\xi\}$ then D can be replaced by $D \cap Q$. Consequently $\operatorname{tr} u(Q \setminus \mathcal{S}_q(u)) = \infty$ and we conclude that $\xi \in \mathcal{S}_{q,0}(u)$. This completes the proof of (6.58). $\square$

Proposition 6.13 *Let F be a $\mathfrak{T}_q$-closed set. Then $\mathcal{S}_q(U_F) = b_q(F)$.*

Proof. Let $\xi \in \mathbb{R}^N$ such that F is $C_{\frac{2}{q},q'}$-thin at ξ. Let Q be a $\mathfrak{T}_q$-open neighborhood of ξ such that $\widetilde{Q} \subset^q F^c$. Then $[U_F]_Q = U_{F\cap\widetilde{Q}} = 0$. Therefore $\xi \in \mathcal{R}_q(U_F)$.

Conversely, assume that $\xi \in F \cap \mathcal{R}_q(U_F)$, thus there exists a $\mathfrak{T}_q$-open neighborhood Q of ξ such that $[U_F]_Q$ is moderate. But $[U_F]_Q = U_{F\cap\widetilde{Q}}$ which implies $C_{\frac{2}{q},q'}(F\cap\widetilde{Q}) = 0$ and $Q \subset \mathcal{R}(u)$. Now, note that $C_{\frac{2}{q},q'}(F) \le C_{\frac{2}{q},q'}(F\cap Q) + C_{\frac{2}{q},q'}(Q^c)$. Thus F is $\mathfrak{T}_q$-thin at ξ. □

6.5 The initial value problem

The following notations will be used in the sequel.

Notation 6.14 *a: We denote by $\mathfrak{M}_+(\mathbb{R}^N)$ the space of positive outer regular Borel measure on $\mathbb{R}^N$.*
b: We denote by $\mathcal{C}_q(\mathbb{R}^N)$ the space of couples (τ, F) such that F is $\mathfrak{T}_q$-closed, $\tau \in \mathfrak{M}_+(\mathbb{R}^N)$, $\mathfrak{T}_q$-supp$(\tau) \subset \widetilde{F^c}$ and $\tau\chi_{F^c}$ is $\mathfrak{T}_q$-locally finite.
c: We denote by $\mathbb{T} : \mathcal{C}_q(\mathbb{R}^N) \to \mathfrak{M}_+(\mathbb{R}^N)$ the mapping given by $\nu = \mathbb{T}(\tau, F)$ where ν is defined as in (6.45) with $\mathcal{R}_q(u)$, $\mathcal{S}_q(u)$ replaced by F, F^c respectively. In this setting ν is the measure representation of the couple (τ, F).
d: If $(\tau, F) \in \mathcal{C}_q(\mathbb{R}^N)$ the set

$$F_\tau = \{\xi \in \mathbb{R}^N : \tau(Q \setminus F) = \infty \quad \forall Q \ \ \mathfrak{T}_q\text{-open neighborhood of } \xi\} \tag{6.60}$$

is called the set of explosion points of τ.

Remark. Note that $F_\tau \subset F$ (because $\tau\chi_{F^c}$ is $\mathfrak{T}_q$-locally finite) and $F_\tau \subset \widetilde{F^c}$ (because τ vanishes outside this set). Thus

$$F_\tau \subset b_q(F^c) \cap F. \tag{6.61}$$

Proposition 6.15 *Let ν be a positive Borel measure on $\mathbb{R}^N$.*
(i) The initial value problem

$$\partial_t u - \Delta u + |u|^{q-1}u = 0, \ \ u > 0 \text{ in } Q_\infty = \mathbb{R}^N \times (0,T), \ \ \mathrm{tr}(u) = \nu \text{ in } \mathbb{R}^N \times \{0\}. \tag{6.62}$$

possesses a solution if and only if $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$.
(ii) Let $(\tau, F) \in \mathcal{C}_q(\mathbb{R}^N)$ and put $\nu := \mathbb{T}(\tau, F)$. Then $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ if and only if

$$\tau \in \mathfrak{M}_q(\mathbb{R}^N), \quad F = b_q(F) \bigcup F_\tau. \tag{6.63}$$

(iii) Let $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ and denote

$$\begin{aligned} \mathcal{E}_\nu &:= \{E : \ E \ \mathfrak{T}_q\text{-quasi-closed}, \ \nu(E \cap K) < \infty, \ \forall \text{ compact } K \subset \mathbb{R}^N\} \\ \mathcal{D}_\nu &:= \{D : \ D \ \mathfrak{T}_q\text{-open}, \widetilde{D} \sim^q E \text{ for some } E \in \mathcal{E}_\nu\}. \end{aligned} \tag{6.64}$$

Then a solution of (6.62) is given by $u = v \oplus U_F$ where

$$G := \bigcup_{\mathcal{D}_\nu} D, \quad F := G^c, \quad v := \sup\{u_\nu \chi_E : \ E \in \mathcal{E}_\nu\}. \tag{6.65}$$

Note that if $E \in \mathcal{E}_\nu$ then $\nu\chi_E$ is locally bounded Borel measure which does not charge sets of $C_{\frac{2}{q},\, q'}$-capacity zero. Recall that if μ is a positive measure possessing these properties, then u_μ denotes the moderate solution with initial trace μ.
(iv) The solution $u = v \oplus U_F$ is σ-moderate and it is the unique solution of problem (6.62) in the class of σ-moderate solutions. Furthermore, u is the largest solution of the problem.

Proof. The proof is similar to the one in [23].
(A) *If* $u \in \mathcal{U}_+(Q_T)$

$$\mathrm{tr}(u) = \nu \Rightarrow \nu \in \mathfrak{M}_q(\mathbb{R}^N). \tag{6.66}$$

By Proposition 6.8, u_{reg} is σ-moderate and $u \approx_{\mathcal{R}_q(u)} u_{reg}$. Therefore

$$\mathrm{tr}(u)\chi_{\mathcal{R}_q(u)} = \mathrm{tr}(u_{reg})\chi_{\mathcal{R}_q(u)}.$$

By Proposition 6.11, $\overline{\mu}_{\mathcal{R}_q} := \mathrm{tr}(u_{reg}) \in \mathfrak{M}_q(\mathbb{R}^N)$. If v is defined as in (6.65) then

$$v = \sup\{[u]_F : \ F \ \ \mathfrak{T}_q\text{-closed } F \subset^q \mathcal{R}_q(u)\} = u_{reg}, \tag{6.67}$$

where the second equality holds by definition. Indeed, by Theorem 6.12, for every $\mathfrak{T}_q$-open set Q, $[u]_Q$ is moderate if and only if $\nu(K \cap \widetilde{Q} \setminus A) < \infty$ for some set A with capacity zero and for any compact K subset of $\mathbb{R}^N$. This means that $[u]_Q$ is moderate if and only if there exists $E \in \mathcal{E}_\nu$ such that $\widetilde{Q} \sim^q E$. When this is the case,

$$\mathrm{tr}([u]_Q) = \mu_{\mathcal{R}_q}(u)\chi_{\widetilde{Q}} = \mu_{\mathcal{R}_q}(u)\chi_E = \nu\chi_E.$$

Thus $v \geq u_{reg}$. On the other hand, if $E \in \mathcal{E}_\nu$, then $\widetilde{E} \subset^q \mathcal{R}_q(u)$ and $\mu_{\mathcal{R}_q}(u)(K \cap \widetilde{E}) = \mu_{\mathcal{R}_q}(u)(K \cap E) < \infty$ for any compact K subset of $\mathbb{R}^N$. Therefore by Proposition 6.8-(ix), $\widetilde{E}$ is regular, i.e, there exist a $\mathfrak{T}_q$-open regular set Q such that $E \subset^q Q$. Hence $u_{\nu\chi_E} \leq [u]_Q$ and we conclude that $v \leq u_{reg}$. This proves (6.67). In addition, if $E \cap \mathcal{S}_q(u) \neq \emptyset$ then $\nu(E) = \infty$, by Definition 6.10. Therefore ν is outer regular with respect to $\mathfrak{T}_q$-topology.

Next we must prove that ν is essentially absolutely continuous. Let Q be a $\mathfrak{T}_q$-open set and A a non-empty $\mathfrak{T}_q$-closed subset of Q such that $C_{\frac{2}{q},\, q'}(A) = 0$. Either $\nu(Q \setminus A) = \infty$, in which case $\nu(Q \setminus A) = \nu(Q)$, or $\nu(Q \setminus A) < \infty$, in which case $Q \setminus A \subset \mathcal{R}_q(u)$ and

$$\nu(Q \setminus A) = \overline{\mu}_{\mathcal{R}_q}(Q \setminus A) = \overline{\mu}_{\mathcal{R}_q}(Q) < \infty.$$

Let $\xi \in A$ let D be a $\mathfrak{T}_q$-open subset of Q such that $\xi \in D \subset \widetilde{D} \subset^q Q$. Let B_n be a $\mathfrak{T}_q$-open neighborhood of $A \cap \widetilde{D}$ such that $C_{\frac{2}{q},\, q'}(B_n) < 2^{-n}$ and $B_n \subset^q D$. Then

$$[u]_D \leq [u]_{E_n} + [u]_{B_n}, \qquad E_n = \widetilde{D} \setminus B_n.$$

Since $\lim_{n\to\infty}[u]_{B_n}=0$ it follows that $[u]_D=[u]_{E_n}$. Now since $E_n\subset\mathcal{R}_q(u)$, $\nu(E_n)\le\nu(Q\setminus A)<\infty$, we have by definition of ν and Proposition 6.8-(ix) that $[u]_{E_n}$ is moderate. Also in view of Lemma 2.8 and Lemma 2.7(ii)-[19], we have for some positive constant C

$$\int_0^T\int_K[u]_{E_n}^q dxdt\le C\nu(E_n)\le C\nu(Q\setminus A)<\infty,$$

for any compact $K\subset\mathbb{R}^N$. Therefore

$$\int_0^T\int_K[u]_D^q dxdt<\infty\qquad\forall\, K\subset\mathbb{R}^N, K\text{ compact.}$$

which implies that $[u]_D$ is moderate and thus $D\subset\mathcal{R}_q(u)$. Since every point A has a neighborhood D as above we conclude that $A\subset\mathcal{R}_q(u)$ and hence $\nu(A)=\mathrm{tr}_{\mathrm{R}}(u)(A)=0$. If A is any a non-empty Borel subset of Q such that $C_{\frac{2}{q},\,q'}(A)=0$, by inequality $C_{\frac{2}{q},\,q'}(\widetilde{A})\subset cC_{\frac{2}{q},\,q'}(A)$, we have that ν is absolutely continuous and $\nu\in\mathfrak{M}_q(\mathbb{R}^N)$.

Secondly we prove:

(B) *Suppose that $(\tau,F)\in\mathcal{C}_q(\mathbb{R}^N)$ satisfies (6.63) and put $\nu=\mathbb{T}(\tau,F)$. Then the solution $u=v\oplus U_F$, with v as in (6.65), satisfies* $\mathrm{tr}(u)=\nu$.

By (6.66), this also implies that $\nu\in\mathfrak{M}_q(\mathbb{R}^N)$.

Clearly v is a σ-moderate solution. The fact that τ is $\mathfrak{T}_q$-locally finite in F^c and essentially absolutely continuous relative to $C_{\frac{2}{q},\,q'}$ implies that

$$G:=F^c\subset\mathcal{R}_q(v),\quad \mathrm{tr}(v)\chi_G=\tau_G. \tag{6.68}$$

It follows from the definition of v that $F_\tau\subset\mathcal{S}_q(v)$. Hence, by Proposition 6.13 and (6.56) we have

$$F=b_q(F)\bigcup F_\tau\subset\mathcal{S}_q(v)\bigcup\mathcal{S}_q(U_F)\subset\mathcal{S}_q(u)\subset F. \tag{6.69}$$

Thus, $\mathcal{S}_q(u)=F$, $v=u_{reg}$ and $\tau=\mathrm{tr}(u_{reg})$. Thus $\mathrm{tr}^c(u)=(\tau,F)$ which is equivalent to $\mathrm{tr}(u)=\nu$.

Next we show: **(C)** *Suppose that $(\tau,F)\in\mathcal{C}_q(\mathbb{R}^N)$ and that there exists a solution u such that* $\mathrm{tr}^c(u)=(\tau,F)$. *Then*

$$\tau=\mathrm{tr}_{\mathcal{R}_q}(u)=\mathrm{tr}(u_{reg}),\quad F=\mathcal{S}_q(u). \tag{6.70}$$

If $U:=u_{reg}\oplus U_F$ then $\mathrm{tr}(U)=\mathrm{tr}(u)$ *and $u\le U$. U is the unique σ-moderate solution of (6.62) and (τ,F) satisfies condition (6.63).* Assertion (6.70) follows by Proposition 6.8-(i) and Definition 6.10. Since u_{reg} is σ-moderate, it follows, by Theorem 6.11, that $\tau\in\mathfrak{M}_q(\mathbb{R}^N)$.

By Proposition 6.8 (vi), $u\approx_{\mathcal{R}_q(u)}u_{reg}$. Therefore $w:=u\ominus u_{reg}$ vanishes on $\mathcal{R}_q(u)$ so that $w\le U_F$. Note that $u-u_{reg}\le w$ and therefore

$$u\le u_{reg}\oplus w\le U. \tag{6.71}$$

By their definitions $\mathcal{S}_{q,0}(u)=F_\tau$ and by Theorem 6.12 (vi) and Proposition 6.13,

$$\begin{aligned}\mathcal{S}_q(U)&=\mathcal{S}_q(u_{reg})\bigcup\mathcal{S}_q(U_F)=\mathcal{S}_q(u_{reg})\bigcup b_q(U_F)\\&=\mathcal{S}_{q,0}(u)\bigcup b_q(U_F)=F_\tau\bigcup b_q(U_F).\end{aligned} \tag{6.72}$$

On the other hand, $\mathcal{R}_q(U) \supset \mathcal{R}_q(u_{\mathcal{R}_q}) = \mathcal{R}_q(u)$ and, as $u \le U$, $\mathcal{R}_q(U) \subset \mathcal{R}_q(u)$. Hence $\mathcal{R}_q(U) = \mathcal{R}_q(u)$ and $\mathcal{S}_q(U) = \mathcal{S}_q(u)$. Therefore, by (6.70) and (6.72), $F = \mathcal{S}_q(U) = F_\tau \cup b_q(U_F)$. Thus (τ, F) satisfies (6.63) and $\mathrm{tr}^c(U) = (\tau, F)$. The fact that U is the maximal solution with this trace follows from (6.71).

The solution U is σ-moderate because both u_{reg} and U_F are σ-moderate solutions (concerning U_F, see Proposition 5.21).

The uniqueness of the solution in the class of σ-moderate solutions follows from Proposition 6.11-(iv).

Finally we prove:

(D) *If $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ then the couple (τ, F) defined by*

$$v := \sup\{u_\nu \chi_E : \; E \in \mathcal{E}_\nu\}, \quad \tau := \mathrm{tr}(v), \quad F = \mathcal{R}_q^c(v), \tag{6.73}$$

satisfies (6.63). This is the unique couple in $\mathcal{C}_q(\mathbb{R}^N)$ satisfying $\nu = \mathbb{T}(\tau, F)$. The solution v is σ-moderate so that $\tau \in \mathfrak{M}_q(\mathbb{R}^N)$.

We claim that $u := v \oplus U_F$ is a solution with initial trace $\mathrm{tr}^c(u) = (\tau, F)$. Indeed $u \ge v$ so that $\mathcal{R}_q(u) \subset \mathcal{R}_q(v)$. On the other hand since τ is $\mathfrak{T}_q$-locally finite in $\mathcal{R}_q(v) = F^c$, it follows that $\mathcal{S}_q(u) \subset F$. Thus $\mathcal{R}_q(v) \subset \mathcal{R}_q(u)$ and we conclude that $\mathcal{R}_q(u) = \mathcal{R}_q(v)$ and $F = \mathcal{S}_q(u)$. This also implies that $v = u_{reg}$.

Finally

$$\mathcal{S}_q(u) = \mathcal{S}_q(v) \bigcup b_q(\mathcal{S}_q(U_F)) = b_q(F) \bigcup F_\tau,$$

so that F satisfies (6.63).

The fact that, for $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$, the couple (τ, F) defined by (6.73) is the only one in $\mathcal{C}_q(\mathbb{R}^N)$ satisfying $\nu = \mathbb{T}(\tau, F)$ follows immediately from the definition of these spaces.

At end, statements **A-D** imply (i)-(iv). $\square$

Remark. If $\nu \in \mathfrak{M}_q(\mathbb{R}^N)$ then G and v as defined by (6.65) have the following alternative representation:

$$G := \bigcup_{\mathcal{E}_\nu} E = \bigcup_{\mathcal{F}_\nu} Q, \qquad v := \sup\{u_\nu \chi_Q : \; Q \in \mathcal{F}_\nu\}, \tag{6.74}$$

$$\mathcal{E}_\nu := \{Q : \; E \; \mathfrak{T}_q\text{-open}, \; \nu(Q \cap K) < \infty, \; \forall \text{ compact } K \subset \mathbb{R}^N\}. \tag{6.75}$$

To verify this remark we first observe that Lemma 2.8 implies that if A is a $\mathfrak{T}_q$-open set then there exists an increasing sequence of $\mathfrak{T}_q$-quasi closed sets $\{E_n\}$ such that $A = \cup_{n=1}^\infty E_n$. In fact, in the notation of Lemma 2.8 (II)(i)-(ii), we may choose $E_n = F_n \setminus L$ where $L = A' \setminus A$, is a set of capacity zero.

Therefore

$$\bigcup_{\mathcal{D}_\nu} D \subset \bigcup_{\mathcal{F}_\nu} Q \subset \bigcup_{\mathcal{E}_\nu} E := H.$$

On the other hand, if $E \in \mathcal{E}_\nu$ then $\mu_{\mathcal{R}_q}(u)(K \cap \widetilde{E}) = \mu_{\mathcal{R}_q}(u)(K \cap E) = \nu(K \cap E) < \infty$, for any compact $K \subset \mathbb{R}^N$. By Proposition 6.8-(ix), $\widetilde{E}$ is regular, i.e., there exists a $\mathfrak{T}_q$-open regular set Q such that $E \subset^q Q$. Thus $H = \bigcup_{\mathcal{D}_\nu} D$.

If D is a $\mathfrak{T}_q$-open regular set then $D = \cup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is an increasing sequence of $\mathfrak{T}_q$-quasi closed sets. We infer

$$u_{\nu\chi_D} = \lim_{n\to\infty} u_{\nu\chi_{E_n}}.$$

Therefore

$$\sup\{u_{\nu\chi_Q} :\ Q \in \mathcal{D}_\nu\} \leq \sup\{u_{\nu\chi_Q} :\ Q \in \mathcal{F}_\nu\} \leq \sup\{u_{\nu\chi_Q} :\ Q \in \mathcal{E}_\nu\}.$$

On the other hand, if $E \in \mathcal{E}_\nu$, there exists a $\mathfrak{T}_q$-open regular set Q such that $E \subset^q Q$. Consequently the equality follows.

7 The equation $\partial_t u - \Delta u + Vu = 0$

Let $0 < T \leq \infty$, $Q_T := \mathbb{R}^N \times (0,T)$, $C > 0$ and $V : Q_T \to [0,\infty)$ be a Borel function satisfying

$$0 \leq V(x,t) \leq \frac{C}{t} \qquad \forall (x,t) \in Q_T. \tag{7.1}$$

In this section we prove a general representation theorem for positive solutions of

$$\partial_t u - \Delta u + Vu = 0 \qquad \text{in } Q_T. \tag{7.2}$$

7.1 Preliminaries

We recall that $\mathfrak{M}(\mathbb{R}^N)$ is the set of Radon measures on $\mathbb{R}^N$ and $\mathfrak{M}_+(\mathbb{R}^N)$ its positive cone.

Definition 7.1 *Let $\mu \in \mathfrak{M}(\mathbb{R}^N)$. We say that u is a weak solution of problem*

$$\begin{aligned} \partial_t u - \Delta u + Vu &= 0 \qquad && \text{in } Q_T \\ u(.,0) &= \mu && \text{in } \mathbb{R}^N, \end{aligned} \tag{7.3}$$

if $u \in L^1_{loc}(\overline{Q}_T)$, $Vu \in L^1_{loc}(\overline{Q}_T)$ and there holds

$$\int\int_{Q_T} u(-\phi_t - \Delta\phi)dxdt + \int\int_{Q_T} Vu\phi dxdt = \int_{\mathbb{R}^N} \phi(x,0)d\mu, \tag{7.4}$$

for all $\phi \in X(Q_T)$, where

$$X(Q_T) = \{\phi \in C_c(\mathbb{R}^N \times [0,T)),\ \phi_t + \Delta\phi \in L^\infty_{loc}(\overline{Q}_T)\}.$$

Remark. The definition implies that for any $\zeta \in C^2_c(\mathbb{R}^N)$, the function $t \mapsto \int \zeta(x)u(x,t)dx$ can be extended by continuity on $[0,T]$ as a continuous function and

$$\lim_{t\to 0} \int_{\mathbb{R}^N} \zeta(x)u(x,t)dx = \int_{\mathbb{R}^N} \zeta d\mu. \tag{7.5}$$

Therefore $\|u(.,t)\|_{L1(\Omega)}$ remains uniformly bounded on $(0,T)$ for any bounded open set $\Omega \subset \mathbb{R}^N$.

Lemma 7.2 *Let $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ and assume that there exists a positive weak solution u of problem (7.3) where V satisfies (7.1). Then for any smooth bounded domain Ω there exists a unique positive weak solution v of problem*

$$\begin{array}{rll} \partial_t v - \Delta v + Vv = 0 & & \text{in } Q_T^\Omega = \Omega \times (0,T) \\ v = 0 & & \text{on } \partial_l Q_T^\Omega = \partial\Omega \times (0,T) \\ v(.,0) = \chi_\Omega \mu & & \text{in } \Omega, \end{array} \tag{7.6}$$

where χ_Ω is the characteristic function on Ω and there holds $v \le u$ in $\Omega \times (0,T)$.

Proof. Let $\{t_j\}$ be a decreasing sequence converging to 0, such that $t_j < T$ for all $j \in \mathbb{N}$. We consider the following problem

$$\begin{array}{rll} \partial_t v - \Delta v + Vv = 0 & & \text{in } \Omega \times (t_j,T) \\ v = 0 & & \text{on } \partial\Omega \times (t_j,T) \\ v(.,t_j) = u(.,t_j) & & \text{in } \Omega. \end{array}$$

Since $u(.,t_j) \in L^1(\Omega)$ and $0 \le V \in L^\infty(\Omega \times (t_j,T])$, there exists a unique positive weak solution v_j of the above problem, smaller than the solution $\mathbb{H}^\Omega[u(.,t_j)\chi_\Omega]$, where $\mathbb{H}^\Omega$ is the heat operator in $Q^\Omega := \Omega \times (0,\infty)$ with zero boundary condition furthermore $v_j \le u$ in $\Omega \times (t_j,T)$ for all $j \in \mathbb{N}$. By standard parabolic estimates we may assume that the sequence $\{v_j\}$ converges locally uniformly in $\Omega \times (0,T]$ to a nonnegative function v smaller than u. If $\phi \in C^{1,1;1}(\overline{Q_T^\Omega})$ vanishes on $\partial_l Q_T^\Omega$ and satisfies $\phi(x,T) = 0$, the following identity holds

$$\int_{t_j}^T \int_\Omega v_j(-\phi_t - \Delta\phi)dxdt + \int_{t_j}^T \int_\Omega V v_j \phi dxdt + \int_\Omega \phi(x,T-t_j)u(x,T-t_j)dx = \int_\Omega \phi(x,0)u(x,t_j)dx,$$

where, in the above equality, we have take as test function $\phi(.,.-t_j)$. It follows by the dominated convergence theorem, that v is a weak solution of problem (7.6). □

Lemma 7.3 *Assume (7.1) holds and let u be a positive weak solution of problem (7.4) with $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$. Then for any $(x,t) \in \mathbb{R}^N \times (0,T]$, we have*

$$\lim_{R\to\infty} u_R = u,$$

where $\{u_R\}$ is the increasing sequence of the weak solutions of the problem (7.6) with $\Omega = B_R(0)$. Moreover, the convergence is uniform in any compact subset of $\mathbb{R}^N \times (0,T]$.

Proof. By the maximum principle (see [19, Remark 2.5]),

$$u_R \le u_{R'} \le u$$

for all $0 < R \le R'$. Thus $u_R \to w \le u$. Also by standard parabolic estimates, this convergence is locally uniformly. Now by dominated convergence theorem, we have that w is a weak solution of problem (7.3) with initial data μ. We set $\widetilde{w} = u - w \ge 0$. Then $\widetilde{w}$ satisfies in the weak sense

$$\begin{array}{rcll} \widetilde{w}_t - \Delta\widetilde{w} + V\widetilde{w} & = & 0 & \text{in } \mathbb{R}^N \times (0,T), \\ w(x,t) & \ge & 0 & \text{in } \mathbb{R}^N \times (0,T) \\ \widetilde{w}(x,0) & = & 0 & \text{in } \mathbb{R}^N. \end{array}$$

Since $\widetilde{w}$ satisfies in the weak sense

$$\begin{aligned} \widetilde{w}_t - \Delta \widetilde{w} &\leq 0 \qquad \text{in } \mathbb{R}^N \times (0,T), \\ w(x,t) &\geq 0 \qquad \text{in } \mathbb{R}^N \times (0,T), \\ \widetilde{w}(x,0) &= 0 \qquad \text{in } \mathbb{R}^N, \end{aligned}$$

We extend $\widetilde{w}$ by 0 for $t \leq 0$, with the same notation and set $\widetilde{w}_n := \widetilde{w} * J_{\epsilon_n}$ where $\{J_{\epsilon_n}\}$ is a sequence of mollifiers in $\mathbb{R}^{N+1}$. Then $\widetilde{w}_n \leq 0$, therefore $\widetilde{w} = 0$. □

Lemma 7.4 *Let $u \in C^{2,1}(\mathbb{R}^N \times (0,T])$ be a positive solution of*

$$\partial_t u - \Delta u + Vu = 0 \qquad \text{in} \qquad \mathbb{R}^N \times (0,T).$$

Assume that, for any $x \in \mathbb{R}^N$, there exists an open bounded neighborhood U of x such that

$$\int_0^T \int_U u(y,t)V(y,t)dxdt < \infty$$

Then $u \in L^1(U \times (0,T))$ and there exists a unique positive Radon measure μ such that

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(y,t)\phi(x)dx = \int_{\mathbb{R}^N} \phi(x)d\mu \qquad \forall \phi \in C_0^\infty(\mathbb{R}^N).$$

Proof. Since $Vu \in L^1(U \times (0,T))$ the following problem has a weak solution v (see [19]).

$$\begin{aligned} \partial_t v - \Delta v &= Vu, \qquad \text{in} \qquad U \times (0,T], \\ v(x,t) &= 0 \qquad \text{on } \partial U \times (0,T] \\ v(x,0) &= 0 \qquad \text{in } U. \end{aligned}$$

Thus the function $w = u + v$ is a positive solution of the heat equation, thus there exists a unique Radon measure μ such that

$$\lim_{t\to 0} \int_U w(y,t)\phi(x)dx = \int_U \phi(x)d\mu, \quad \forall \phi \in C_0^\infty(U).$$

But the initial data of v is zero, thus the result follows by a partition of unity and Lemma 7.3.□

7.2 Representation formula for positive solutions

Assume V satisfies (7.1) in Q_T and let u be a positive solution of (7.2). If $\psi \in C^{2,1}(\mathbb{R}^N \times (0,T])$, we set $u(x,t) = e^{\psi(x,t)}v(x,t)$. Then v satisfies

$$\partial_t v - \Delta v - 2\nabla\psi\nabla v - |\nabla\psi|^2 v - 2\Delta\psi v + (\psi_t + \Delta\psi + V)\, v = 0 \text{ in } \qquad \mathbb{R}^N \times (0,T]. \tag{7.7}$$

We choose ψ to be the solution of the problem

$$\begin{aligned} -\psi_t - \Delta\psi &= V \qquad \text{in} \qquad \mathbb{R}^N \times (0,T] \\ \psi(x,T) &= 0 \qquad \text{in} \qquad \mathbb{R}^N. \end{aligned}$$

Then

$$\psi(t,x)=\int_t^T\int_{\mathbb{R}^N}\frac{1}{(4\pi(s-t))^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4(s-t)}}V(x,s)dxds. \tag{7.8}$$

By a straightforward calculation we verify that
1. $0\le\psi\le C\ln\frac{T}{t}$,
2. $|\nabla\psi|\le C_1(T)+C_2(\ln\frac{T}{t})$.
Thus (7.7) becomes

$$\partial_t v-\Delta v-\sum_{i=1}^n(2\psi_{x_i}v)_{x_i}-|\nabla\psi|^2v=0.$$

Since $\int_0^1|\ln t|^pdt<\infty$ for all $p\ge 1$, we verify by 1 and 2 that

$$\int_0^T\sup_{x\in\mathbb{R}^N}|\psi|^qds<M_1<\infty\qquad\forall q\ge 1$$

and

$$\int_0^T\sup_{x\in\mathbb{R}^N}|\nabla\psi|^qds<M_2<\infty\qquad\forall q\ge 1.$$

For $A_{i,j}=\delta_{ij}$, $A_i=2\psi_{x_i}$ $B_i=0$ and $C=|\nabla\psi|^2$ we see that the above operator satisfies the condition H in [3] for $R_0=\infty$ and $p=\infty$. Thus there exists a kernel $\Gamma(x,t;y,s)$, defined in $Q_T\times Q_T$ satisfying the estimates

$$C_1(T,n,M_2)\frac{1}{(4\pi(t-s))^{\frac{n}{2}}}e^{-A_1\frac{|x-y|^2}{4(t-s)}}\le\Gamma(x,t;y,s)\le C_2(T,n,M_2)\frac{1}{(4\pi(t-s))^{\frac{n}{2}}}e^{-A_2\frac{|x-y|^2}{4(t-s)}}, \tag{7.9}$$

where $A_1,\ A_2>0$ depend on $T,\ n,\ M_2$ with the property that v admits the following representation formula:

$$v(x,t)=\int_{\mathbb{R}^N}\Gamma(x,t;y,0)d\mu(y), \tag{7.10}$$

where μ is a uniquely defined positive Radon measure on $\mathbb{R}^N$, and there holds

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Gamma(x,t;y,0)\phi(x)d\mu(y)dx=\int_{\mathbb{R}^N}\phi d\mu\qquad\forall\phi\in C_0^\infty(\mathbb{R}^N).$$

Furthermore, if $e^{-\gamma|x|^2}u_0\in L^2(\mathbb{R}^N)$ for some $\gamma\ge 0$, and if u_0 is continuous at y, then

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\Gamma(x,t;y,0)u_0(x)dx=u_0(y). \tag{7.11}$$

Finally we have

$$u(x,t)=e^{\psi(x,t)}\int_{\mathbb{R}^N}\Gamma(x,t;y,0)d\mu(y). \tag{7.12}$$

8 σ-moderate solutions

8.1 Preliminaries

Proposition 8.1 *Let $u \in \mathcal{U}_+(Q_T)$. Then*

$$\max(u_{\mathcal{R}_q}, [u]_{\mathcal{S}_q(u)}) \le u \le u_{reg} + [u]_{\mathcal{S}_q(u)}. \tag{8.1}$$

Proof. The principle of the proof is similar as the one in [16].
By Proposition 6.8-(vii), the function $v = u \ominus u_{reg}$ vanishes on $\mathcal{R}_q(u)$ i.e., $\mathfrak{T}_q$-supp $(v) \subset \mathcal{S}_q(u)$. Thus v is a solution dominated by u and supported in $\mathcal{S}_q(u)$, which implies that $v \le [u]_{\mathcal{S}_q(u)}$ by Definition 5.27. Since $u - u_{reg} \le v$ this implies the inequality on the right hand side of (8.1). The inequality on the left hand side is obvious. $\square$

Proposition 8.2 *Let $u \in \mathcal{U}_+(Q_T)$ and let A, B be two disjoint $\mathfrak{T}_q$-closed subsets of $\mathbb{R}^N$. If $\mathfrak{T}_q$-supp $(u) \subset A \cup B$ and $[u]_A$, $[u]_B$ are σ-moderate then u is σ-moderate. Furthermore*

$$u = [u]_A \oplus [u]_B = [u]_A \vee [u]_B. \tag{8.2}$$

Proof. The proof is same as in [16].
By Proposition 6.11-(iii) there exist two increasing sequences $\{\tau_n\}$, $\{\tau'_n\} \subset W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ such that

$$u_{\tau_n} \uparrow [u]_A, \qquad u_{\tau'_n} \uparrow [u]_B.$$

By proposition 5.26, $\mathfrak{T}_q$-supp $(\tau_n) \subset^q A$ and $\mathfrak{T}_q$-supp $(\tau'_n) \subset^q B$. Thus $C_{\frac{2}{q},\, q'}\left(\mathfrak{T}_q\text{-supp}(\tau_n) \cap \mathfrak{T}_q\text{-supp}(\tau'_n)\right) = 0$, and

$$u_{\tau_n} \vee u_{\tau'_n} = u_{t_n} \oplus u_{t'_n} = u_{\tau_n + \tau'_n}.$$

By (5.34) and Definition 5.27,

$$\max([u]_A, [u]_B) \le u \le [u]_A + [u]_B. \tag{8.3}$$

Therefore,

$$\max(u_{\tau_n}, u'_{\tau_n}) \le u \Rightarrow u_{\tau_n + \tau'_n} \le u.$$

On the other hand

$$u - u_{\tau_n + \tau'_n} \le [u]_A - u_{\tau_n} + [u]_B - u_{\tau'_n} \downarrow 0.$$

Thus

$$\lim_{n \to \infty} u_{\tau_n + \tau'_n} = u, \tag{8.4}$$

so that u is σ- moderate.
The assertion (8.2) is equivalent to the statements: (a) u is the largest solution dominated by $[u]_A + [u]_B$ and (b) u is the smallest solution dominating $\max([u]_A, [u]_B)$. Let

$$u \le w := [u]_A \oplus [u]_B \le [u]_A + [u]_B.$$

Thus we have $[u]_A \leq [w]_A$. But $[w]_A \leq w \leq [u]_A + [u]_B \;\Rightarrow\; [w]_A - [u]_A \leq [u]_B$. By Notation 3.3 we have

$$v = [([w]_A - [u]_A)_+]_\dagger \leq [u]_B, \qquad v \leq [w]_A,$$

that is

$$\mathfrak{T}_q\text{-supp}\,(v) \subset A \qquad \text{and} \qquad \mathfrak{T}_q\text{-supp}\,(v) \subset B.$$

But $A \cap B = \emptyset$, which implies $v = 0$ and $[w]_A \leq [u]_A$. Similarly, we have $[w]_B \leq [u]_B$. Thus

$$[w]_A = [u]_A, \qquad [w]_B \leq [u]_B.$$

By (8.3) and the fact that for any Borel E $[u]_E \leq [u]_{\widetilde{E}\cap A} + [u]_{\widetilde{E}\cap B}$, there holds

$$\mathcal{S}_q(u) = \mathcal{S}_q(w).$$

Let Q be a $\mathfrak{T}_q$-open regular set in $\mathcal{R}_q(w)$, then $Q \in \mathcal{R}_q(u)$. Using (5.34), (5.35) and the fact that $\mathfrak{T}_q$-supp $(w) \subset A \cap B$, we derive

$$[w]_Q \leq [w]_{\widetilde{Q}\cap A} + [w]_{\widetilde{E}\cap B} = [[w]_A]_{\widetilde{Q}} + [[w]_B]_{\widetilde{Q}} = [u]_{\widetilde{Q}\cap A} + [u]_{\widetilde{Q}\cap B}.$$

Since $[w]_Q,\ [u]_Q$ are moderate solutions and $A \cap B = \emptyset$, we have $[u]_{\widetilde{Q}\cap A} \oplus [u]_{\widetilde{Q}\cap B} \leq [u]_Q$, which implies $[w]_Q = [u]_Q$. Thus by Proposition 6.8-(ii) $w_{\mathcal{R}_q} = u_{\mathcal{R}_q}$, and since u is σ-moderate by Proposition 6.15 and the remark below we get

$$u \leq w \leq u_{\mathcal{R}_q} \oplus U_F.$$

By the uniqueness of σ- moderate solutions (Theorem 6.11-(iv)), w and u coincide. This proves (a).
For the statement (b), we note that

$$u_{\tau_n + \tau'_n} = u_{\tau_n} \vee u_{\tau'_n} \leq [u]_A \vee [u]_B,$$

since $u_{\tau_n} \leq [u]_A$ and $u_{\tau'_n} \leq [u]_B$. Thus the result follows by (8.4) and (8.3), by letting n tend to infinity. $\square$

8.2 Characterization of positive solutions of $\partial_t u - \Delta u + u^q = 0$

The following notation is used throughout the subsection.
Let $u \in \mathcal{U}_+(Q_T)$. Set

$$V = u^{q-1},$$

then

$$V \leq \left(\frac{1}{q-1}\right)^{q-1} t^{-1}.$$

Thus $u \in C^{2,1}(\mathbb{R}^N \times (0,T]$ and satisfies

$$\partial_t u - \Delta u + Vu = 0, \qquad \text{in} \qquad \mathbb{R}^N \times (0,1]. \tag{8.5}$$

Hence, by the representation formula (7.12), u satisfies

$$u(x,t) = e^{\psi} \int_{\mathbb{R}^N} \Gamma(x,t;y,0) d\mu(y), \qquad \forall\, t \leq T, \tag{8.6}$$

where μ is Radon measure (see subsection 7.2). The measure μ is called the *extended initial trace* of u.
For any Borel set E set

$$\mu_E = \mu \chi_E \qquad \text{and} \qquad (u)_E = e^{\psi} \int_{\mathbb{R}^N} \Gamma(x,t;y,0) d\mu_E, \qquad \forall\, t \leq T.$$

Lemma 8.3 *Let F be a compact subset of $\mathbb{R}^N$. Then*

$$(u)_F \leq [u]_F, \qquad \forall t \leq T.$$

Proof. We follow the ideas of [16], adapted to the parabolic framework.
Let A be a Borel subset of $\mathbb{R}^N$. Let $0 < \beta \leq \frac{T}{2}$ and let v_β^A be the positive solution of

$$\begin{array}{ll} \partial_t v - \Delta v + V v = 0 & \text{in } \mathbb{R}^N \times (\beta, T] \\ v(.,\beta) = u(.,\beta)\chi_A(.) & \text{in } \mathbb{R}^N. \end{array} \tag{8.7}$$

Also let w_β^A be the positive solution of

$$\begin{array}{ll} \partial_t w - \Delta w + |w|^{q-1} w = 0 & \text{in } \mathbb{R}^N \times (\beta, T] \\ w(.,\beta) = \chi_A(.) u(.,\beta) & \text{in } \mathbb{R}^N. \end{array}$$

Then by the maximum principle $w_\beta^A \leq u$, which implies

$$0 = \frac{dw_\beta^A}{dt} - \Delta w_\beta^A + (w_\beta^A)^q \leq \frac{dw_\beta^A}{dt} - \Delta w_\beta^A + V w_\beta^A.$$

Thus w_β^A is a supersolution of (8.7), and by the maximum principle (see [3] or Lemma 7.3), we have

$$v_\beta^A \leq w_\beta^A \leq u.$$

For any sequence $\{\beta_k\}$ decreasing to zero one can extract a subsequence $\{\beta_{k_n}\}$ such that $\{w_{\beta_{k_n}}^A\}$ and $\{v_{\beta_{k_n}}^A\}$ converge locally uniformly; we denote the limits w^A and v^A respectively (the limits may depend on the sequence). Then $w^A \in \mathcal{U}_+(Q_T)$ while v^A is a solution of (8.5), and

$$v^A \leq w^A \leq [u]_{\widetilde{Q}}, \qquad \forall Q \text{ open}, A \subset Q. \tag{8.8}$$

The second inequality follows from the fact that $\mathfrak{T}_q$-supp$\,(w_\beta^A) \subset \widetilde{Q}$ for any β.

Now we set $v^A_{\beta_{k_n}} = e^{\psi}\widetilde{v}_n$, where ψ is the function in subsection 7.2. Then $\widetilde{v}_n$ is the solution of

$$\begin{array}{rcl}\partial_t v - \Delta v - 2\nabla\psi\nabla v - |\nabla\psi|^2 v - 2\Delta\psi v + (\psi_t + \Delta\psi + V)\, v = 0 & \text{in} & \mathbb{R}^N \times (\beta_{k_n}, T]. \\ v(.,\beta_{k_n}) = \chi_A(.) \displaystyle\int_{\mathbb{R}^N} \Gamma(.,\beta_{k_n}; y, 0) d\mu(y) & \text{in} & \mathbb{R}^N.\end{array}$$

Now using the representation formula in [3], we derive that for any open $Q \supset A$, there holds

$$\begin{aligned}\widetilde{v}_n(x,t) &= \int_{\mathbb{R}^N} \chi_A(x)\Gamma(x, t-\beta_{k_n}; y, 0) \left(\int_{\mathbb{R}^N} \Gamma(x,\beta_{k_n}; y, 0) d\mu(y)\right) dx \\ &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \chi_A(x)\Gamma(x, t-\beta_{k_n}; y, 0)\Gamma(x,\beta_{k_n}; y, 0) dx\right) dx d\mu(y) \\ &\leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \chi_Q(x)\Gamma(x, t-\beta_{k_n}; y, 0)\Gamma(x,\beta_{k_n}; y, 0) dx\right) dx d\mu(y).\end{aligned}$$

Therefore, by (7.11), estimate (7.9) and using the fact that $\Gamma(x, t-s; y, 0)$ is a continuous function for any $s < t$ (see [3]), we can let $k_n \to \infty$ in the above inequality and get

$$\lim_{n\to\infty} \widetilde{v}_n \leq \int_{\mathbb{R}^N} \Gamma(x,t;y,0) d\mu_{\widetilde{Q}}.$$

Hence

$$v^A \leq (u)_{\widetilde{Q}}.$$

We apply the same procedure to the set A^c extracting a further subsequence of $\{\beta_{k_n}\}$ in order to obtain the limits v^{A^c} and w^{A^c}. Thus

$$v^{A^c} \leq w^{A^c} \leq [u]_{\widetilde{Q'}}, \qquad \forall Q' \text{ open}, A^c \subset Q'.$$

Note that

$$v^A + v^{A^c} = u, \quad v^A \leq (u)_{\widetilde{Q}}, \quad v^{A^c} \leq (u)_{\widetilde{Q'}}.$$

Therefore

$$v^A = u - v^{A^c} \geq (u)_{(\widetilde{Q'})^c}. \tag{8.9}$$

Now, given F compact, let A be a closed set and O an open set such that $F \subset O \subset A$. Note that $A^c \cap F = \emptyset$. By (8.9) with $Q' = A^c$

$$v^A \geq (u)_O.$$

By (8.8)

$$v^A \leq w^A \leq [u]_{\widetilde{Q}} \qquad \forall Q \text{ open}, A \subset Q,$$

and consequently

$$(u)_F \leq (u)_O \leq [u]_{\widetilde{Q}}. \tag{8.10}$$

By Lemma 2.8, we can choose a sequence of open sets $\{Q_n\}$ such that $\cap\widetilde{Q}_j = E' \sim^q F$, thus by Proposition 5.24-(iii) $[u]_{Q_j} \downarrow [u]_F$. The result follows by (8.10). $\square$

In the next lemma we prove that the extended initial trace of a positive solution of (3.1) is absolutely continuous with respect to the $C_{\frac{2}{q},q'}$-capacity.

Lemma 8.4 *Let $u \in \mathcal{U}_+(Q_T)$, μ its extended initial trace. If E is a Borel set and $C_{\frac{2}{q},q'}(E) = 0$ then $\mu(E) = 0$.*

Proof. The proof is similar as the one in [16]. If F is a compact subset of E, then $C_{\frac{2}{q},q'}(F) = 0$ and therefore by Proposition 5.17, $U_F = 0$. But $[u]_F = u \wedge U_F = 0$. Therefore, by Lemma 8.3 $(u)_F = 0$. Consequently $\mu(F) = 0$. As this holds for every compact subset of E we conclude that $\mu(E) = 0$. □

We recall that, if $\nu \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$, then for any $T > 0$, there exists a constant $C > 0$ independent on ν (see Lemma 2.11-[22]) such that

$$C^{-1}||\nu||_{W^{-\frac{2}{q},q}(\mathbb{R}^N)} \leq ||\mathbb{H}[\nu]||_{L^q(Q_T)} \leq C||\nu||_{W^{-\frac{2}{q},q}(\mathbb{R}^N)}, \tag{8.11}$$

where $\mathbb{H}[\nu]$ is the solution of the heat equation in Q_∞ with ν as initial data.

Lemma 8.5 *Let $u \in \mathcal{U}_+(Q_T)$, μ its extended initial trace and $\nu \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}_+^b(\mathbb{R}^N)$. Suppose that there exists no positive solution of (3.1) dominated by the supersolution $v = \inf\{u, \mathbb{H}[\nu]\}$. Then $\mu \perp \nu$.*

Proof. Set $V' = v^{q-1}$, then v is a supersolution of

$$\partial_t w - \Delta w + V'w = 0 \qquad \text{in} \qquad \mathbb{R}^N \times (0,T]. \tag{8.12}$$

We first claim that there exists no positive solution of the above problem dominated by v. We proceed by contradiction in assuming that there exists a positive solution $w \leq v$ of (8.12). Then w satisfies

$$\partial_t w - \Delta w + w^q \leq \partial_t w - \Delta w + V'w = 0.$$

Since

$$\|v\|_{L^q(Q_T)} \leq \|\mathbb{H}[\nu]\|_{L^q(Q_T)} \approx \|\nu\|_{W^{-\frac{2}{q},q}(\mathbb{R}^N)}$$

this implies that w is a positive moderate solution of (3.1) dominated by v, contrary to assumption. Now for any $t \leq T$, we have by representation formula (7.12),

$$\begin{aligned}
\inf\{u, \mathbb{H}[\nu]\} &= \inf\left\{e^{\psi}\int_{\mathbb{R}^N}\Gamma(x,t;y,0)d\mu(y), \mathbb{H}[\nu]\right\} \\
&\geq \inf\left\{\int_{\mathbb{R}^N}\Gamma(x,t;y,0)d\mu(y), \mathbb{H}[\nu]\right\} \\
&\geq C\inf\left\{\mathbb{H}[\mu](\frac{t}{A_2},x), \mathbb{H}[\nu](t,x)\right\} \\
&\geq C\inf\left\{\mathbb{H}[\mu](\frac{t}{\max(A_2,1)},x), \mathbb{H}[\nu](\frac{t}{\max(A_2,1)},x)\right\},
\end{aligned}$$

where, in the above inequalities, we have used estimates (7.9) and the constants $C > 0$, $A_2 > 0$ therein.

Now since $\inf\left\{\mathbb{H}[\mu](\frac{t}{\max(A_2,1)},x),\mathbb{H}[\nu](\frac{t}{\max(A_2,1)},x)\right\}$ is a supersolution of $\partial_t w - \frac{1}{\max(A_2,1)}\Delta w = 0$, there exists a positive Radon measure $\widetilde{\nu}$ such that

$$\lim_{t\to 0}\int_{\mathbb{R}^N}\phi(x)\inf\left\{\mathbb{H}[\mu](\frac{t}{\max(A_2,1)},x),\mathbb{H}[\nu](\frac{t}{\max(A_2,1)},x)\right\}dx = \int_{\mathbb{R}^N}\phi(x)d\widetilde{\nu} \quad \forall\phi\in C_0^\infty(\mathbb{R}^N).$$

Thus in view of Lemmas 7.3 and 7.4, there exists a positive weak solution $\widetilde{v}\le v$ of the problem

$$\begin{array}{rcll} \partial_t w - \Delta w + V'w &=& 0 & \text{in} \quad \mathbb{R}^N\times(0,T]. \\ w(.,0) &=& \widetilde{\nu} & \text{in} \quad \mathbb{R}^N, \end{array}$$

and by the first claim it yields $\widetilde{\nu}=0$.

By the Lebesgue-Radon-Nikodym Theorem we can write $d\nu = \phi d\mu + d\sigma$, where $0\le\phi\in L^1_{loc}(\mathbb{R}^N,\mu)$ and $\sigma\perp\mu$. Thus we have

$$\begin{array}{rcl} 0 &=& \lim_{t\to 0}\int_{\mathbb{R}^N}\phi(x)\inf\left\{\mathbb{H}[\mu](\frac{t}{\max(A_2,1)},x),\mathbb{H}[\nu](\frac{t}{\max(A_2,1)},x)\right\}dx \\ &\ge& \lim_{t\to 0}\int_{\mathbb{R}^N}\phi(x)h(\frac{t}{\max(A_2,1)},x,y)\min\{f,1\}(y)d\mu(y)dx \\ &=& \lim_{t\to 0}\int_{\mathbb{R}^N}\phi(y)\min\{f,1\}(y)d\mu(y) = 0, \end{array}$$

where, we recall it, $h(t,x,y)$ in the heat kernel in Q_∞. Hence $f=0$ and $\nu\perp\mu$. $\square$

Lemma 8.6 *Let $u\in\mathcal{U}_+(Q_T)$, μ its extended initial trace and suppose that for every $\nu\in\mathfrak{M}^b_+(\mathbb{R}^N)\cap W^{-\frac{2}{q},q}(\mathbb{R}^N)$ there exists no positive solution of (3.1) dominated by $v=\inf(u,\mathbb{H}[\nu])$. Then $u=0$.*

Proof. The proof is similar as the one in [16]. By Lemma 8.5,

$$\mu\perp\nu \qquad \forall\nu\in W^{-\frac{2}{q},q}(\mathbb{R}^N)\cap\mathfrak{M}^b_+(\mathbb{R}^N).$$

Suppose that $\mu\ne 0$. By Lemma 8.4, μ vanishes on sets of $C_{\frac{2}{q},q'}$-capacity zero. Thus, there exists an increasing sequence $\{\nu_k\}\subset W^{-\frac{2}{q},q}(\mathbb{R}^N)\cap\mathfrak{M}^b_+(\mathbb{R}^N)$ which converges to μ. Thus $\mu\perp\nu_k$ and for every $k\in\mathbb{N}$ there exists a Borel set $A_k\subset\mathbb{R}^N$ such that

$$\mu(A_k)=0 \text{ and } \nu_k(A_k^c)=0.$$

Therefore, if $A=\cup_k A_k$ then

$$\mu(A)=0 \text{ and } \nu_k(A^c)=0 \quad \forall k.$$

Since $\nu_k\le\mu$ we have $\nu_k(A)=0$ and therefore $\nu_k=0$. Contradiction. $\square$

Lemma 8.7 *Let $u\in\mathcal{U}_+(Q_T)$. Then $[u]_{\mathcal{S}_q(u)}$ is σ-moderate.*

Proof. To simplify the notations we put $u_{\mathcal{S}} = [u]_{\mathcal{S}_q(u)}$ and denote $F := \mathfrak{T}_q\text{-supp}\,(u_{\mathcal{S}})$. Incidentally, $F \subset \mathcal{S}_q(u)$; since if $\mathcal{S}_q(u)$ is thin at ξ, then $\mathcal{S}_q(u)^c \cup \{\xi\}$ is $\mathfrak{T}_q$-open and $\mathcal{S}_q(u)^c \cup \{\xi\} \sim^q \mathcal{S}_q(u)^c$. Thus by definition of F, we see that F consists precisely of the $C_{\frac{2}{q},\,q'}$-thick points of $\mathcal{S}_q(u)$. The set $\mathcal{S}_q(u) \setminus F$ is contained in the singular set of $u_{\mathcal{R}_q}$.

For $\nu \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ we denote by u_ν the solution of (3.1) with initial trace ν. Put

$$u^* := \sup\{u_\nu :\ \nu \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N),\ u_\nu \le u_{\mathcal{S}}\}. \tag{8.13}$$

By Lemma 8.6 the family over which the supremum is taken is not empty. Therefore u^* is a positive solution of (3.1) and, by Proposition 6.11-(iii), it is σ-moderate, thus it is the largest σ-moderate solution dominated by $u_{\mathcal{S}}$. We denote by $\{\nu_n\} \subset W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ an increasing sequence such that $u^* = \lim_n n \to \infty u_{\nu_n}$.

Let $F^* = \mathfrak{T}_q\text{-supp}\,(u^*)$. Then F^* is $\mathfrak{T}_q$-closed and $F^* \subset F$. Suppose that

$$C_{\frac{2}{q},\,q'}(F \setminus F^*) > 0.$$

Then there exists a compact set $E \subset F \setminus F^*$ such that $C_{\frac{2}{q},\,q'}(E) > 0$ and $(F^*)^c =: Q^*$ is a $\mathfrak{T}_q$-open set containing E. Furthermore by Lemma 2.7 there exists a $\mathfrak{T}_q$-open set Q' such that $E \subset^q Q \subset \widetilde{Q'} \subset^q Q^*$. Since $Q' \subset^q \mathfrak{T}_q\text{-supp}\,(u_{\mathcal{S}})$, $[u_{\mathcal{S}}]_{Q'} > 0$ and therefore by Lemma 8.6, there exists a positive measure $\tau \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ supported in $\widetilde{Q'}$ such that $u_\tau \le u_{\mathcal{S}}$. As the $\mathfrak{T}_q\text{-supp}\,(\tau)$ is a $\mathfrak{T}_q$-closed set disjoint from F^*, it follows that the inequality $u^* \ge u_\tau$ does not hold. On the other hand, since $\tau \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ and $u_\tau \le u_{\mathcal{S}}$, it follows that $u_\tau \le u^*$. This contradiction shows that

$$C_{\frac{2}{q},\,q'}(F \setminus F^*) = 0. \tag{8.14}$$

Since $u_{\nu_n} \uparrow u^*$, $\mathfrak{T}_q\text{-supp}\,(u_{\nu_n}) \subset \mathfrak{T}_q\text{-supp}\,(u^*) := F^*$. Therefore there exists a $\mathfrak{T}_q$-closed set $F_0^* \subset F^*$ such that $\mathcal{S}_q(u^*) = F_0^*$ and $\mathcal{R}_q(u^*) = (F_0^*)^c$. Suppose that

$$C_{\frac{2}{q},\,q'}(F \setminus F_0^*) > 0.$$

Let Q' be a $\mathfrak{T}_q$-open subset of $\mathcal{R}_q(u^*)$ such that $[u_{\mathcal{S}}]_{Q'}$ is a moderate solution, then $\widetilde{Q'} \subset^q \mathcal{R}_q(u^*)$ and $[u^*]_{\widetilde{Q'}}$ is a moderate solution of (3.1), i.e.,

$$\int_0^T \int_{\mathbb{R}^N} [u^*]^q_{\widetilde{Q'}} \phi(x)\,dxdt < \infty \qquad \forall \phi \in C_0(\mathbb{R}^N),\ \phi \ge 0.$$

On the other hand Q' is a $\mathfrak{T}_q$-open subset of $F = \mathfrak{T}_q\text{-supp}\,(u_{\mathcal{S}})$; therefore the initial trace of $[u^*]_{\widetilde{Q'}}$ has no regular part, i.e.,

$$\mathcal{R}_q([u^*]_{\widetilde{Q'}}) = 0 \text{ and } \mathcal{S}_q([u^*]_{\widetilde{Q'}}) = \mathfrak{T}_q\text{-supp}\,([u^*]_{\widetilde{Q'})};$$

we say that $[u^*]_{\widetilde{Q'}}$ is *a purely singular solution of (3.1)*. It follows that $v := \Big[[u_{\mathcal{S}}]_{\widetilde{Q'}} - [u^*]_{\widetilde{Q'}}\Big]_\dagger$ is a purely singular solution of (3.1).

Let v^* be defined as in (8.13) with u replaced by v. Then v^* is a singular σ-moderate solution of (3.1). Since v^* is smaller than u and since it is σ-moderate it is dominated by u^*. On the other hand, since v^* is singular and $\mathfrak{T}_q$-supp $(v^*) \subset^q \widetilde{Q'} \subset^q \mathcal{R}_q(u^*)$ it follows that u^* is not larger or equal to v^*, i.e. $(v^* - u^*)_+$ is not identically zero. Since both u^* and v^* are σ-moderate, it implies that there exists $\tau \in W^{-\frac{2}{q},q}(\mathbb{R}^N) \cap \mathfrak{M}^b_+(\mathbb{R}^N)$ such that $u_\tau \leq v^*$, and $(u_\tau - u^*)$ is not identically zero. Therefore $u^* \lneq \max(u^*, u_\tau)$. The function $\max(u^*, u_\tau)$ is a subsolution of (3.1) and the smallest solution above it, denoted by Z, is strictly larger than u^*. However $u_\tau \leq v^* \leq u^*$ and consequently $Z = u^*$.

This contradiction proves that $C_{\frac{2}{q},\, q'}(Q') = 0$, for any set $Q' \subset \mathcal{R}_q(u^*)$ such that $[u]_{Q'}$ is moderate solution, that is $C_{\frac{2}{q},\, q'}(\mathcal{R}_q(u^*)) = 0$ which implies

$$C_{\frac{2}{q},\, q'}(F \setminus F_0^*) = 0. \tag{8.15}$$

In conclusion, u^* is σ-moderate, $\mathfrak{T}_q$-supp $(u^*) \subset F$ and $F_0^* = \mathcal{S}_q(u^*) \sim^q F$. Therefore, by Proposition 6.15 and the remark below, $u^* = U_F$. Since by definition $u^* \leq u \leq U_F$, it follows $u^* = u$. □

Theorem 8.8 *Every positive solution of (3.1) is σ-moderate.*

Proof. We borrow the ideas of the proof to [16]. By Proposition 6.8-(i), $\mathcal{R}_q(u)$ has regular decomposition $\{Q_n\}$. Furthermore

$$v_n = [u]_{Q_n} \uparrow u_{\mathcal{R}_q}.$$

Thus the solution $u_{\mathcal{R}_q}$ is σ-moderate and

$$u \ominus u_{\mathcal{R}_q} \leq [u]_{\mathcal{S}_q(u)}.$$

Put

$$u_n = v_n \oplus [u]_{\mathcal{S}_q(u)}.$$

By Lemma 8.7 we have that $[u]_{\mathcal{S}_q(u)}$ is σ-moderate solution, thus by Proposition 8.2, as $\widetilde{Q}_n \cap \mathcal{S}_q(u) = \emptyset$, it follows that u_n is σ-moderate. As $\{u_n\}$ is increasing it follows that $\overline{u} = \lim_{n\to\infty} u_n$ is a σ-moderate solution of (3.1). In addition

$$v_n \vee [u]_{\mathcal{S}_q(u)} = u_n = v_n \oplus [u]_{\mathcal{S}_q}(u) \Rightarrow \ \max(u_{\mathcal{R}_q}, [u]_{\mathcal{S}_q(u)}) \leq \overline{u} \leq u_{\mathcal{R}_q} + [u]_{\mathcal{S}_q(u)}.$$

This further implies that $\mathcal{S}_q(u) = \mathcal{S}_q(\overline{u})$. By construction we have

$$[u]_{Q_n} = v_n \leq [\overline{u}]_{Q_n}$$

Letting $n \to \infty$ we have by Proposition 6.8

$$u_{\mathcal{R}_q} \leq \overline{u}_{\mathcal{R}_q} \Rightarrow u_{\mathcal{R}_q} = \overline{u}_{\mathcal{R}_q},$$

thus $\mathrm{tr}(u) = \mathrm{tr}(\overline{u})$ and since $\overline{u} \leq u$, we have by Proposition 6.15 and the uniqueness of σ-moderate solutions that $u = \overline{u}$. □

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